A New Solution for Confined–Unconfined Flow Toward a Fully Penetrating Well in a Confined Aquifer

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Abstract

Transient confined-unconfined flow conversion caused by pumping in a confined aquifer (i.e., piezometric head drops below the top confined layer) is complicated, partly due to different hydraulic properties between confined and unconfined regions. For understanding mechanism of the transient confined-unconfined conversion, this paper develops a new analytical solution for the transient confined-unconfined flow toward a fully penetrating well in a confined aquifer. The analytical solution is used to investigate the impacts on drawdown simulation by differences of hydraulic properties, including transmissivity, storativity, and diffusivity defined as a ratio of transmissivity and storativity, between the confined and unconfined regions. It is found that neglecting the transmissivity difference may give an overestimation of drawdown. Instead, neglecting the diffusivity difference may lead to an underestimation of drawdown. The shape of drawdown-time curve is sensitive to the change of storativity ratio, S/S_y, between the confined and unconfined regions. With a series of drawdown data from pumping tests, the analytical solution can also be used to inversely estimate following parameters related to the transient confined-unconfined conversion: radial distance of conversion interface, diffusivity, and specific yield of the unconfined region. It is concluded that using constant transmissivity and diffusivity in theory can result in biased estimates of radial distance of the conversion interface and specific yield of the unconfined region in practice. The analytical solution is useful to gain insight about various factors related to the transient confined-unconfined conversion and can be used for the design of mine drainage and groundwater management in the mining area.

Nomenclature

b	aquifer thickness, [m];
h	initial head, [m];
$h_1(r,t)$	elevation of piezometric surface in unconfined region, [m];
$h_2(r,t)$	elevation of piezometric surface in confined region, [m];
$h^{'}(r_1,t)$	elevation of piezometric surface in observation well, [m];
h_w	elevation of piezometric surface in pumping well, [m];
$\overline{h_1}$	effective elevation of piezometric surface within the range from b to zero in unconfined region, [m];
Q	constant pumping rate, [m^3/s];
K_r	horizontal conductivity of the aquifer, [m/s];
r	radial distance, [m];
r_1	distance of observation well from pumping well, [m];
R	radial distance of conversion interface from pumping well, [m];
S	storativity;
S_s	specific storage, [m^{-1}];
S_y	specific yield of unconfined region;
S/S_y	storativity ratio between the confined and the unconfined regions;
t	time, [t];

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Article impact statement: The paper presents a discussion of effects of differences of hydraulic properties on drawdown during confined-unconfined conversion.

Received April 2017, accepted February 2018.

doi: 10.1111/gwat.12642

t_c	conversion time in pumping well;
$\overline{T} = K_r \overline{h_1}$	variable transmissivity of the unconfined region, [m^2/s];
$T = K_r b$	constant transmissivity of the confined region, [m^2/s];
$\overline{T}/S$	diffusivity of the unconfined region, [m^2/s];
$W(u)$	Theis well function.

Introduction

A transient confined-unconfined flow conversion can occur in a confined aquifer due to heavy pumping, for example, in the processes of mine dewatering and groundwater overexploration (Springer and Bair 1992; Stoner 1981; Walton 1964; Chen et al. 2006). When pumping rate and/or duration in a confined aquifer is sufficiently large, piezometric surface can drop below the confined layer at the vicinity of the pumping well. This creates an unconfined region in the confined aquifer, and the conversion interface between the confined and unconfined regions gradually moves away from the pumping well as the pumping continues.

The pumping-induced flow has been found to be seriously influenced by hydraulic properties of the pumped aquifer (i.e., Wang and Zhan 2009; Hu and Chen 2008; Neuman 1974, 1972). Generally, the transient confined-unconfined conversion can result in differences of hydraulic properties (e.g., transmissivity, storativity, and diffusivity) between the confined and unconfined regions. This study is to investigate such differences of hydraulic properties on the transient confined-unconfined flow. For several decades, there has been ongoing debate on how the differences of hydraulic properties affect the transient confined-unconfined flow behaviors in literature of numerical and analytical modeling of aquifer tests. For numerical solutions, Rushton and Wedderburn (1971) employed a resistance-capacitance electrical analogue to analyze the confined-unconfined conversion behavior in aquifers. A specific yield for the unconfined region was used to replace the storativity of the confined aquifer in the numerical solution. Based on the Dupuit's assumptions, Elango and Swaminathan (1980) simulated the confined-unconfined flow using the finite element methods with four-sided mixed-curved isoperimetric elements; however, it was limited to analyzing a steady-state flow. Wang and Zhan (2009) presented a seminumerical solution for the transient confined-unconfined flow, and the solution considered the differences of both transmissivity and storativity during the confined-unconfined conversion. These numerical models solved the system's governing equations using numerical techniques, which made it difficult to understand mechanisms of the transient confined-unconfined flow.

This study is focused on analytical solutions to gain insights on pumping-induced flow. Assuming that drawdown in the unconfined region is considerably smaller than thickness of the confined aquifer, Moench and Prickett (1972) developed the first analytical solution (hereinafter referred to as MP model) for the transient confined-unconfined flow by using a constant transmissivity. They also investigated the effect of the different storativity in the confined and unconfined regions. However, the use of the constant transmissivity is invalid if the drawdown in the unconfined region is sufficiently large. Hu and Chen (2008) considered a constant diffusivity in term of the Girinskii's potential which was defined as a potential of a steady-state groundwater flow in a layered porous medium (Bear 1972). For the transient confined-unconfined conversion, Hu and Chen (2008) reported an analytical approximate solution by using the Theis equation. This solution was improved from the work of Chen (1983), which is thus referred to as Chen model hereinafter. Theoretically speaking, the use of a constant diffusivity during the confined-unconfined conversion was doubtful due to the fact that the diffusivity of the unconfined region must be much smaller than that of the confined region (Kruseman and Ridderna 1991).

Our literature review suggests that, for the existing analytical solutions, analyses of the differences of hydraulic properties between the confined and the unconfined regions have been insufficient. The MP model, which is the only choice in a widely-used software for aquifer tests namely AQTESOLV (Duffield 2007), neglects the different transmissivity. The Chen model, which is the most recent analytical solution in literature, does not account for the difference of diffusivity. Moreover, using the aforementioned theoretically-limited analytical solutions can result in biased estimates of drawdown-time curves and parameters related to the transient confined-unconfined conversion in the pumped confined aquifer. However, this problem has not received sufficient attention in literature. Therefore, it is necessary to derive more theoretically advanced analytical solution to provide a comprehensive understanding on the effects of the differences of hydraulic properties and to avoid biased estimate of drawdown-time curve and aquifer parameters in practice.

In this paper, a new analytical solution for the transient confined-unconfined flow induced by pumping in a fully penetrating confined aquifer is developed. The flow in the confined region is described by two-dimensional differential equation of the seepage system, and the flow in the unconfined region is governed by the Boussinesq equation associated with boundary conditions that account for flow continuity in the conversion interface. The Boussinesq equation is linearized by a new approach that links different transmissivity, storativity, and diffusivity between the confined and the unconfined regions. After deriving the analytical solution using the Boltzmann transform, the solution is used to develop a new practical approach for assessing the dynamic development of the unconfined region and for estimating the diffusivity and specific yield of the unconfined region. The effects of the different transmissivity, storativity, and diffusivity between the confined and unconfined regions on drawdown simulation and aquifer parameter estimates are further discussed by comparing the results obtained using the proposed analytical solution with those obtained using the former analytical solutions (the MP model and the Chen model) and numerical solutions given by Visual-MODFLOW in synthetic case studies.

Conceptual-Mathematical Model

Consider an infinite nonleaky confined aquifer that extends horizontally and has a horizontal initial piezometric head, h, (Figure 1). The confined aquifer is homogeneous and isotropic, and is fully penetrated by a pumping well with a constant pumping rate, Q. An observation well with an infinitesimal diameter is located at the distance of r_1 from the pumping well.

The groundwater flow that occurs after the pumping but before the confined-unconfined flow conversion can be described by the Theis solution. When hydraulic head, $h_1(r,t)$, is lower than the confined layer, that is, $0 \le h_1(r,t) \le b$, where b is the thickness of the confined aquifer, an unconfined region is created in the area with a radial distance, r, that is smaller than R, that is, $0 < r \le R$ (Figure 1), where R is the radial distance of the transient confined-unconfined interface from the pumping well and r is the radial distance from the pumping well . After the unconfined region appears, the transient confined-unconfined flow can be described by the Boussinesq equation

$$\frac{K_r}{r}\frac{\partial}{\partial r}\left(rh_1\frac{\partial h_1}{\partial r}\right) = S_y\frac{\partial h_1}{\partial t} \tag{1a}$$

where K_r is the horizontal conductivity of the aquifer, S_y is the specific yield in the unconfined region, and t is the pumping time.

It is noted that the Boussinesq equation was proposed based on the Dupuit assumption in which the hydraulic gradient was assumed to be sufficiently small and thus the vertical flow component could be eliminated, that is, $v = v_r = -K_r \sin\theta = -K_r \tan\theta,\ \theta < 30^\circ$ (Figure 1) (Bear 1972), where v is the flow velocity, v_r is the horizontal flow velocity component. For the practical purpose, a relationship is given as $\tan\theta < \tan\theta_1 < \tan 30^\circ$ in Figure 1, that is, $\tan\theta < \frac{h' - h_w}{r_1} < 0.58$, where h' is elevation of the piezometric surface in the observation well, h_w is elevation of the piezometric surface in pumping well, r_1 is the distance between the pumping and the observation well, and $\tan\theta_1 = \frac{h' - h_w}{r_1}$. Generally, the Boussinesq equation is valid if r_1 is sufficiently large, that is, $r_1 > 1.72b$. If $r_1 < 1.72b$, the use of Boussinesq equation can be accepted when $h' < 0.58\ r_1 + h_w$.

The boundary condition representing the fully penetrating well is

$$\lim_{r\to 0} 2\pi K_r h_1 r \frac{\partial h_1}{\partial r} = Q \tag{1b}$$

where Q is the pumping rate. In the confined region ($b \le h_2(r,t) \le h,\ r \ge R$), the transient flow is governed by

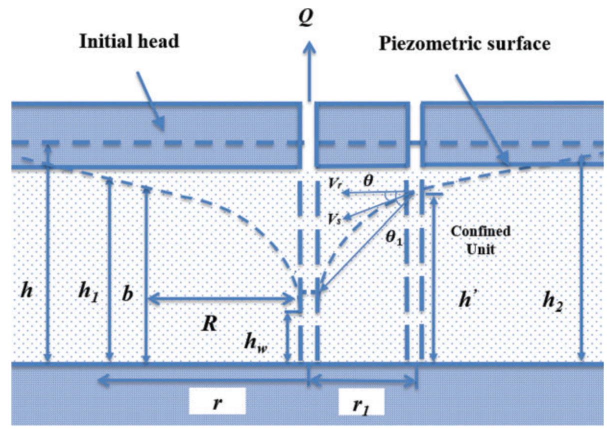

Figure 1. A schematic diagram of the transient confined-unconfined flow toward a fully penetrating well of infinitesimal diameter in an infinite confined aquifer.

$$\frac{K_r}{r}\frac{\partial}{\partial r}\left(rb\frac{\partial h_2}{\partial r}\right) = S\frac{\partial h_2}{\partial t} \tag{2a}$$

where $h_2(r,t)$ is the elevation of the piezometric surface measured with respect to the aquifer base in the confined region, and S is the storativity of the confined aquifer. The far-field boundary condition is

$$h_2\,(r \to \infty, t) = h \tag{2b}$$

where h is the initial piezometric head. The initial boundary for both the confined and the unconfined flow is

$$h_1\,(r,0) = h_2\,(r,0) = h \tag{3}$$

The conversion interface between the confined and unconfined regions moves away from pumping well as the pumping continues. The boundary conditions at the conversion interface for the confined and unconfined flow are

$$\frac{\partial h_1\,(R,t)}{\partial r} = \frac{\partial h_2\,(R,t)}{\partial r} \tag{4a}$$

$$h_1\,(R,t) = h_2\,(R,t) = b \tag{4b}$$

Derivation of Analytical Solutions

Based on the above mathematical model of the governing equation and the boundary conditions, the analytical solutions of the flow in the confined and the unconfined conditions are presented below.

Analytical Solution for Transient Unconfined Flow

The Boussinesq equation is linearized by using the following relation (Bear 1972),

$$\frac{\partial h_1}{\partial t} = \frac{1}{2\overline{h}_1}\frac{\partial\left(h_1^2\right)}{\partial t} \tag{5}$$

Different from other linearization methods assuming that $\overline{h_1}$ is a constant elevation of piezometric surface in the unconfined region (e.g., Bear 1972; Liang and Zhang 2011), $\overline{h_1}$ in Equation 5 is defined as a variable elevation of piezometric surface. In practice, the value of $\overline{h_1}$ ranges from b to 0 in the unconfined region at the time point of interest if $r_1 > 1.72b$. If $r_1 < 1.72b$, the range of $\overline{h_1}$ value is from b to 0.58 r_1.

Substituting Equation 5 to Equations 1a, 1b, 3, and 4b gives

$$\frac{\partial^2 \left(h_1^2\right)}{\partial r^2} + \frac{1}{r}\frac{\partial \left(h_1^2\right)}{\partial r} = \frac{S_y}{K_r \overline{h_1}}\frac{\partial \left(h_1^2\right)}{\partial t} \tag{6a}$$

$$\lim_{r\to 0} \pi K_r r \frac{\partial \left(h_1^2\right)}{\partial r} = Q \tag{6b}$$

$$h_1^2 (R, t) = b^2 \tag{6c}$$

$$h_1^2 (r, 0) = h^2 \tag{6d}$$

By introducing $s_1^{'} (r, t) = b^2 - h_1 (r, t)^2$, Equations 6a-6d become

$$\frac{\partial^2 s_1^{'}}{\partial r^2} + \frac{1}{r}\frac{\partial s_1^{'}}{\partial r} = \frac{S_y}{K_r \overline{h_1}}\frac{\partial s_1^{'}}{\partial t} \tag{7a}$$

$$\lim_{r\to o} \pi K_r r \frac{\partial s_1^{'}}{\partial r} = -Q \tag{7b}$$

$$s_1^{'} (R, t) = 0 \tag{7c}$$

$$s_1^{'} (r, 0) = b^2 - h^2 \tag{7d}$$

Following the derivation given in Appendix S1, Supporting information, the analytical solution to Equations 7a-7d of the transient unconfined flow is

$$\begin{aligned} h_1 (r, t)^2 &= b^2 - \frac{Q}{2\pi K_r}\left[W\left(\frac{S_y r^2}{4K_r \overline{h_1} t}\right) - W\left(\frac{S_y R^2}{4K_r \overline{h_1} t}\right)\right] \\ &= b^2 - \frac{Q}{2\pi K_r}\left[W\left(\frac{S_y r^2}{4\overline{T} t}\right) - W\left(\frac{S_y R^2}{4\overline{T} t}\right)\right] \end{aligned} \tag{8}$$

where $\overline{T} = K_r \overline{h_1}$ is the transmissivity in the unconfined region, and $W(u)$ is the Theis well function. It is noted that the transmissivity in the unconfined region is variable due to the change of the elevation of piezometric surface ($\overline{h_1}$) with time.

Analytical Solution for Transient Confined Flow

By introducing $s_2^{'} (r, t) = h - h_2 (r, t)$, Equations 2a-2b and 4b are rewritten as

$$\frac{\partial^2 s_2^{'}}{\partial r^2} + \frac{1}{r}\frac{\partial s_2^{'}}{\partial r} = \frac{S}{K_r b}\frac{\partial s_2^{'}}{\partial t} \tag{9a}$$

$$s_2^{'} (r \to \infty, t) = 0 \tag{9b}$$

$$s_2^{'} (R, t) = h - b \tag{9c}$$

Following the derivation given in Appendix S2, the analytical solution to Equations 9a-9c for the transient confined flow is

$$\begin{aligned} h_2 (r, t) &= h - \frac{Q}{4\pi K_r b}\frac{\exp\left(-\frac{S_y R^2}{4K_r \overline{h_1} t}\right)}{\exp\left(-\frac{SR^2}{4K_r bt}\right)} W\left(\frac{Sr^2}{4K_r bt}\right) \\ &= h - \frac{Q}{4\pi T}\frac{\exp\left(-\frac{S_y R^2}{4\overline{T} t}\right)}{\exp\left(-\frac{SR^2}{4Tt}\right)} W\left(\frac{Sr^2}{4Tt}\right) \end{aligned} \tag{10}$$

where $T = K_r b$ is the constant transmissivity of the confined region.

Simulation Drawdown

One of the important purposes of mathematical modeling for pumping test is to simulate drawdown-time curve of well flow induced by pumping. In this section, a practical approach using the developed analytical solution is outlined for the drawdown simulation of the transient confined-unconfined flow.

For using the developed analytical solution of Equations 8 and 10 for drawdown simulation, the parameters are generally assumed to be known. The parameters are the constant pumping rate (Q), the initial hydraulic head (h), the thickness (b), and the hydraulic properties (S_s, S_y, and K_r) of the confined aquifer. The other two parameters of a variable elevation of piezometric surface in the unconfined region ($\overline{h_1}$) and the radial distance of the transient confined-unconfined interface from the pumping well (R) are calculated. Hence, the procedure of drawdown simulation for the transient confined-unconfined flow is as follows: (1) calculate the $\overline{h_1}$ and R values; (2) assess the drawdown at the given time by using Equation 10 in the confined region ($R \le r_1$); and (3) evaluate the drawdown-time data by using Equation 8 in the unconfined region ($R > r_1$).

Based on the definition of $\overline{T} = K_r \overline{h_1}$, the expressions on the flow toward the confined-unconfined conversion interface are given by subjecting Equation 10 to the boundary condition (Equation 4b) as

$$b = h - \frac{Q}{4\pi T}\frac{\exp\left(-\frac{S_y R^2}{4K_r \overline{h_1} t}\right)}{\exp\left(-\frac{SR^2}{4Tt}\right)} W\left(\frac{SR^2}{4Tt}\right) \tag{11}$$

We adopted the conceptual model of Hu and Chen (2008) that the groundwater discharge to the pumping well during the pumping duration is equal to the sum of groundwater storage changes in both the unconfined (V_1) and the confined (V_2) regions. As shown in Figure 2, the amount of groundwater pumped from the unconfined region is

$$V_1 = S_y \int_0^R 2\pi r\,(b - h_1)\,\mathrm{d}r = S_y \int_0^R 2\pi r \left\{ b - \sqrt{b^2 - \frac{Q}{2\pi K_r}\left[W\left(\frac{S_y r^2}{4K_r\overline{h_1}t}\right) - W\left(\frac{S_y R^2}{4K_r\overline{h_1}t}\right)\right]} \right\} \mathrm{d}r \qquad (12a)$$

and the amount of groundwater pumped from the confined region is

$$V_2 = S\left[\pi R^2 (h - b) + \int_R^\infty 2\pi r\,(h - h_2)\,\mathrm{d}r\right] = S\pi R^2 (h - b) + S\int_R^\infty r\frac{Q}{2T}\frac{\exp\left(-\frac{S_y R^2}{4K_r\overline{h_1}t}\right)}{\exp\left(-\frac{SR^2}{4Tt}\right)} W\left(\frac{Sr^2}{4Tt}\right)\mathrm{d}r \qquad (12b)$$

The law of conservation of mass gives

$$Qt = V_1 + V_2 \qquad (12c)$$

Subjecting Equations 12a and 12b to Equation 12c leads to

$$Qt = S_y \int_0^R 2\pi r \left\{ b - \sqrt{b^2 - \frac{Q}{2\pi K_r}\left[W\left(\frac{S_y r^2}{4K_r\overline{h_1}t}\right) - W\left(\frac{S_y R^2}{4K_r\overline{h_1}t}\right)\right]} \right\} \mathrm{d}r + S\pi R^2 (h - b) + S\int_R^\infty r\frac{Q}{2T}\frac{\exp\left(-\frac{S_y R^2}{4K_r\overline{h_1}t}\right)}{\exp\left(-\frac{SR^2}{4Tt}\right)} W\left(\frac{Sr^2}{4Tt}\right)\mathrm{d}r \qquad (13)$$

Since Equations 11 and 13 have two unknowns, $\overline{h_1}$ and R, the two unknowns can be determined by solving Equations 11 and 13 simultaneously.

Assess Dynamic Development of Unconfined Region and Estimate Aquifer Parameters by Drawdown Data

Investigation of the parameters related to transient confined-unconfined conversion based on limited drawdown data is the other important purpose of mathematical modeling for the transient confined-unconfined flow. Using the analytical solution mentioned above, comprehensive approaches are developed to assess dynamic development of the unconfined region, diffusivity, and specific yield of unconfined region. The derivation below assumes that the values of parameters, Q, h, b, S_s, and K_r, are known.

Dynamic Development of Unconfined Region

The dynamic development of the unconfined region is generally represented in term of the radial distance of conversion interface from the pumping well, R, (Figure 1). It is necessary to note that Equations 10 and 13 are incapable of direct determining the R value because the value of specific yield, S_y, is unknown.

Denote the elevations of the piezometric surface in the pumping well and the observation well as h_w and $h'(r_1, t)$, respectively, for time t during the pumping test. As shown in Figure 1, r_1 is the distance between the observation and the pumping wells. When h_w is smaller than b, the transient confined-unconfined conversion occurs. The distance (R) of the conversion interface from the pumping well can be determined separately for confined and unconfined conditions based on the analytical solution above. If $h'(r_1, t) \geq b$, the flow toward the observation well is under the confined condition. Therefore, $h'(r_1, t)$ is obtained directly from the analytical solution given in Equation 10 for the confined region as

$$h'(r_1, t) = h - \frac{Q}{4\pi T}\frac{\exp\left(-\frac{S_y R^2}{4Tt}\right)}{\exp\left(-\frac{SR^2}{4Tt}\right)} W\left(\frac{Sr_1^2}{4Tt}\right) \qquad (14a)$$

On the other hand, considering the boundary condition of Equation 4b, Equation 10 at the boundary of the conversion interface between the confined and unconfined flow becomes

$$b = h - \frac{Q}{4\pi T}\frac{\exp\left(-\frac{S_y R^2}{4Tt}\right)}{\exp\left(-\frac{SR^2}{4Tt}\right)} W\left(\frac{SR^2}{4Tt}\right) \qquad (14b)$$

Rearranging Equations 14a and 14b and taking the ratio gives

$$\frac{h - h'(r_1, t)}{h - b} = \frac{W\left(\frac{Sr_1^2}{4Tt}\right)}{W\left(\frac{SR^2}{4Tt}\right)} \qquad (15)$$

The expression of R is the solution of Equation 15. If $h'(r_1, t) < b$, the flow toward the observation well is under the unconfined condition. Therefore, $h^{'}(r_1, t)$ is obtained directly from the analytical solution given in Equation 8 for the unconfined region as

$$h^{'}(r_1, t)^2 = b^2 - \frac{Q}{2\pi K_r}\left[W\left(\frac{S_y r_1^2}{4\overline{T}t}\right) - W\left(\frac{S_y R^2}{4\overline{T}t}\right)\right] \tag{16a}$$

Substituting the expression

$$\frac{S_y}{4\overline{T}t} = -\ln\left[\frac{(h-b)\,4\pi T}{QW\left(\frac{SR^2}{4Tt}\right)}\right]\mathrm{X}\frac{1}{R^2} + \frac{S}{4Tt} \tag{16b}$$

into Equation 16a yields

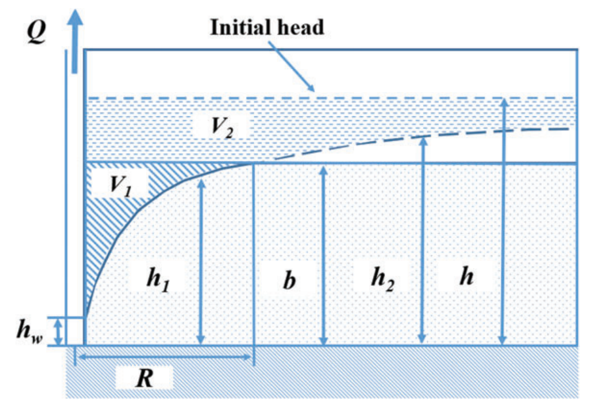

Figure 2. Change of groundwater storage during the confined-unconfined conversion (after Hu and Chen 2008).

$$h^{'}(r_1, t)^2 = b^2 - \frac{Q}{2\pi K_r}\left\{W\left\{-\ln\left[\frac{(h-b)\,4\pi T}{QW\left(\frac{SR^2}{4Tt}\right)}\right]X\frac{r_1^2}{R^2} + \frac{Sr_1^2}{4Tt}\right\} - W\left\{-\ln\left[\frac{(h-b)\,4\pi T}{QW\left(\frac{SR^2}{4Tt}\right)}\right] + \frac{SR^2}{4Tt}\right\}\right\} \tag{17}$$

In the unconfined region, the R value in Equation 17 can thus be simulated for different time points of interest during pumping.

Diffusivity and Specific Yield of Unconfined Region

Based on Equation 11, the diffusivity $(K_r\overline{h_1}/S_y)$ of the unconfined region can be derived directly as

$$\frac{K_r\overline{h_1}}{S_y} = \left[-\ln\left[\frac{(h-b)\,4\pi T}{QW\left(\frac{SR^2}{4Tt}\right)}\right]\mathrm{X}\frac{4t}{R^2} + \frac{S}{T}\right]^{-1} \tag{18}$$

The specific yield (S_y) of the unconfined region can be estimated by relating it with the amount of water discharged to the pumping well. After re-expressing Equation 13 as follows,

$$Qt = S_y\int_0^R 2\pi r\left\{b - \sqrt{b^2 - \frac{Q}{2\pi K_r}\left[W\left(\frac{S_y r^2}{4K_r\overline{h_1}t}\right) - W\left(\frac{S_y R^2}{4K_r\overline{h_1}t}\right)\right]}\right\}\mathrm{d}r + S\pi R^2(h-b) + S\int_0^R r\frac{Q}{2T}\frac{\exp\left(-\frac{S_yR^2}{4K_r\overline{h_1}t}\right)}{\exp\left(-\frac{SR^2}{4Tt}\right)}W\left(\frac{Sr^2}{4Tt}\right)\mathrm{d}r \tag{19}$$

The two unknowns, $\overline{h_1}$ and S_y, can be determined by solving Equations 18 and 19 simultaneously. Accordingly, steps for the use of the proposed analytical solution for parameters evaluation in practice are suggested as: (1) measure drawdown during pumping test; (2) calculate R with Equations 15 and 17 in the confined and the unconfined regions, respectively; and (3) calculate $\overline{h_1}$ and S_y of the unconfined region.

Discussion

In this section, the acceptability of the use of the proposed analytical solution for drawdown simulation and assessments of the parameters related to the transient confined-unconfined flow is investigated in three hypothetical cases, including *Case I*, *Case II*, and *Case III*, of pumping tests. Furthermore, the effect of the differences of transmissivity, storativity, and diffusivity between the confined and the unconfined regions on the drawdown estimates is also discussed via case studies.

In reality, how to readily and accurately find the solutions of Equations 11 and 13, 17, and 18 and 19 is crucial for the practical use of the proposed solution for drawdown simulation and parameters assessment. To our knowledge, the software namely Mathematica (Mureşan 2017) gradually becomes one of the most powerful techniques for math programming, which is thus widely used in the field of research on hydrogeology (Cheng and Sidauruk 1996; Pérez Guerrero and Skaggs 2010; Haneberg 2013; Baalousha 2015; Azarafza et al. 2017). Hence, for automated operation, the solutions of Equations 11 and 13, 17, and 18 and 19 are obtained by using Mathematica in the following three hypothetical cases.

Drawdown-Time Curve of Transient Confined-Unconfined Flow

The drawdown-time curve for the transient confined-unconfined flow is presented in *Case I*. An emphasis is to verify the acceptability of the use of the proposed

Table 1
Given Parameters of the Hypothesized Pumping Tests

Parameters	Value
h (m)	36
b (m)	30
Kr (m/s)	0.0000695
S_s	0.000002
Q (m^3/s)	0.026
Pumping duration (days)	10,000
r_1 *(m)*	*Case I*: 29 *Case II*: 10 *Case III*: 10
S_y	*Case I*: 0. 3 *Case II*: 0.00006, 0.0006, 0.006, and 0. 3 *Case III*: 0.3

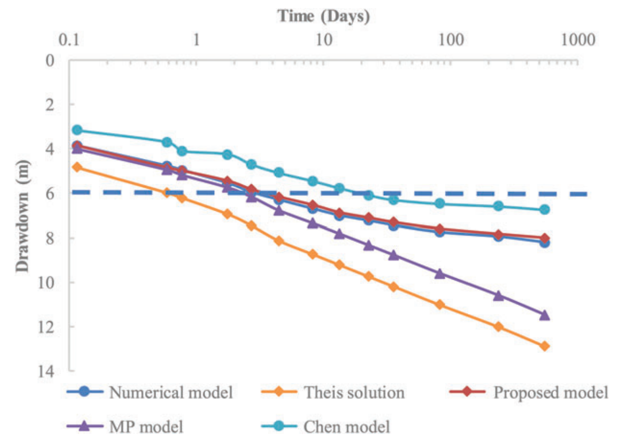

Figure 3. Drawdown-time curves of the transient confined-unconfined flow induced by pumping by the use of different solutions with $S_y = 0.3$ in *Case I*. The radial distance of observation well from the pumping well is 29 m. The confined-unconfined conversion in the observation well happens if drawdown is greater than 6 m.

solution for drawdown simulation during the confine-unconfined flow, which can be further used to investigate the effect of the different transmissivities and diffusivities between the confined and unconfined regions. The given parameter values of *Case I* are listed in Table 1. It notes that the S value of the confined region can be calculated as $S = S_s b$, where S_s is the specific storage of the confined aquifer. The drawdown data evaluated by the proposed analytical solution are compared with those obtained using Visual-MODFLOW and the conventional of MP model and Chen model, which are the special cases of the analytical solution of this study as shown in Appendix S3. The numerical solution for the pumping test in an infinite confined aquifer in *Case I* is constructed by Visual-MODFLOW. It involves one-layer model with a domain size of 5 km × 5 km, 400 rows, 400 columns, and 16×10^4 cells. The pumping well is located at the point with the coordinate of $x = 2520$ and $y = 2530$ and the observation well is situated in the point with the coordinate of $x = 2520$ and $y = 2559$.

Figure 3 compares the drawdown-time curves of the transient confined-unconfined flow with S_y value of 0.3 obtained by different solutions with the parameter values listed in Table 1. It demonstrates that the drawdown of the transient confined-unconfined flow is generally smaller than that of the Theis solution at any given time. The drawdown-time curve of the analytical solution of this study is visually identical to that obtained using Visual-MODFLOW. The use of the constant diffusivity of the Chen model leads to an underestimation of the drawdown and an overestimation of the conversion time at the observation well (Figure 3). Using a constant transmissivity in the MP model, the drawdown values during the entire pumping period are larger than those from the Visual-MODFLOW and the errors are clearly accumulated as pumping continues (Figure 3 and Table S1).

Consequently, the slope of drawdown curve by the proposed solution is essentially identical to that from the Visual-MODFLOW, verifying that the use of the proposed solution for drawdown simulation in the pumping well during the confined-unconfined conversion is accepted in practice. Otherwise, the use of MP model and Chen model for drawdown simulation of the transient confined-unconfined flow must be cautious due to that serious errors can be introduced by using the constant transmissivity (MP model) and diffusivity (Chen model) during the transient confined-unconfined conversion.

Effects of Different Storativity Ratios on Drawdown Simulation

Case II is used to investigate the effect of the different storativity ratios (S/S_y) on drawdown simulation. As mentioned in *Case I*, it is necessary to notice that the assumption of the use of the constant transmissivity (MP model) and diffusivity (Chen model) during the transient confined-unconfined conversion can result in the disability of the MP model and the Chen model in simulation of the drawdown-time curve affected by the different storativity ratios. The acceptability of the proposed solution for the drawdown simulation of the transient confined-unconfined flow has been verified by the comparative analysis with the results from Visual-MODFLOW in *Case I*. Hence, to avoid work duplication, the drawdown data in *Case II* is only assessed by using the proposed solution.

The given parameter values of *Case II* are listed in Table 1. Figure 4 shows the drawdown-time curves of the transient confined-unconfined flow at the observation well with $r_1 = 10$ m with S/S_y values of 0.002, 0.01, 0.1, and 1 between the confined and the unconfined regions, respectively. The conversion happens if the drawdown is greater than 6 m in the case. The figure reveals that the shape of drawdown curve is significantly influenced by the storativity ratio. The increase of S/S_y value enlarges the slopes of the drawdown-time curves at a given time during the confined flow pattern, and shortens the conversion time. If the flow toward the observation well is under the unconfined condition, the slopes of the drawdown-time

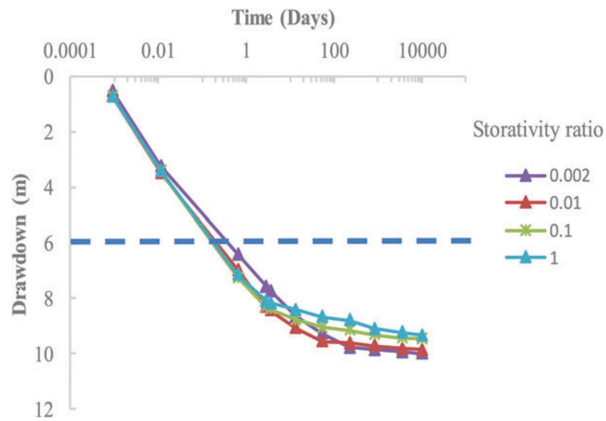

Figure 4. Drawdown-time curves of the transient confined-unconfined flow with different S/S_y values by the proposed solution in *case II*. The confined-unconfined conversion in the observation well happens if drawdown is greater than 6 m in this case.

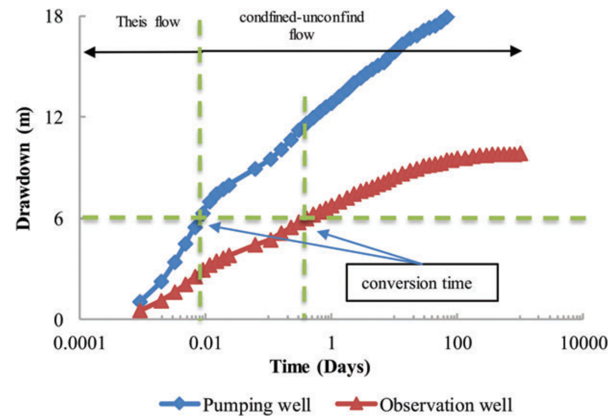

Figure 5. Drawdown-time curves of the pumping and the observation wells by using Visual-MODFLOW in *Case III*. The times of the confined-unconfined conversion in the pumping and the observation wells are 0.0085 and 0.39 days, respectively, when the drawdown in the pumping and the observation wells becomes greater than 6 m.

curves at the late time may decrease with the increase of S/S_y value.

Effects of Different Hydraulic Properties on Parameter Estimates

Case III is presented to investigate the use of the proposed solution, the MP model, and the Chen model for estimates of parameters related to the confined-unconfined conversion with known drawdown in practice. The parameters include the radial distance of conversion interface, the diffusivity, and the specific yield of the unconfined region.

Generally speaking, the known drawdown data at the pumping well in such case can be assessed by both the proposed analytical solution and the numerical model of Visual-MODFLOW as mentioned in *Case I*. However, in order to ensure the rationality of analysis results, the drawdown-time curves at the pumping well and the observation well in the hypothetical pumping test are only simulated by Visual-MODFLOW (Figure 5) with the parameter values listed in Table 1. The construction of the numerical model by Visual-MODFLOW and the location of the pumping well are the same as those in *Case I*. The observation well is situated in the point with the coordinate of $x = 2520$ and $y = 2540$. The effective elevation of the piezometric surface in the pumping well for the validity of the Boussinesq equation ranges from 30 to 5.8 m in the *Case III*.

The confined-unconfined conversion occurs at 0.0085 days when the drawdown in the pumping well becomes greater than 6 m. After that, the confined-unconfined flow dominates as pumping proceeds and the conversion time in the observation well with $r_1 = 10$ m is 0.39 days.

Figure 6 plots the radial distance, R, from the pumping well to the conversion interface with time. The R values obtained using the analytical solutions (Equations 15 and 17) of this study are visually identical to those obtained using Visual-MODFLOW over the entire

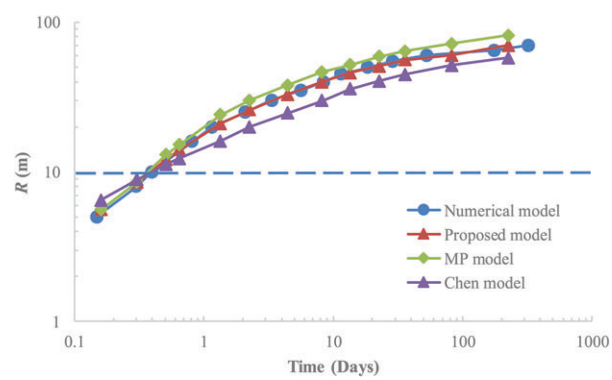

Figure 6. Radial distance of conversion interface from the pumping well, *R*, by different analytical solutions and MODFLOW solutions.

observation period. The R values of the flow in the unconfined condition obtained using the MP model are larger than those obtained using Visual-MODFLOW. The errors are accumulative as the pumping continues. As shown in Table S2, the largest difference is more than 12 m. It suggests that the use of the constant transmissivity can lead to serious error to R values estimation when the drawdown is sufficiently large. For the Chen model, its corresponding R values are larger than the numerical solutions obtained using Visual-MODFLOW, when the flow toward the observation well is under the confined condition. After the conversion time, the R values of the Chen model become smaller than the solutions obtained using Visual-MODFLOW, suggesting an underestimation of R. The errors are found to be introduced by the use of the average spatial diffusivity in the Chen model at a given time during the transient confined-unconfined conversion.

The specific yield, S_y, of the unconfined region can be investigated by the proposed analytical solution (Equations 18 and 19). The results shown in Figure 7 indicate that the S_y value of the analytical solution is a

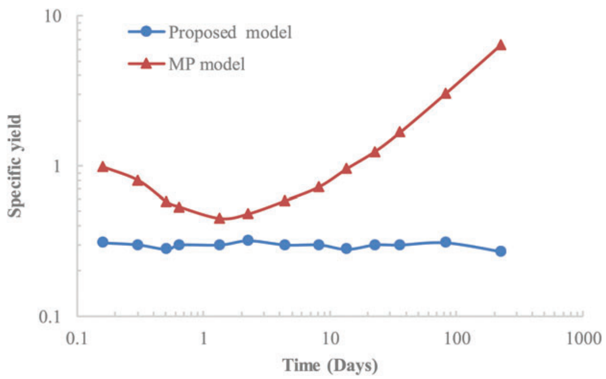

Figure 7. Plots of S_y values versus normal time, indicating that the use of constant transmissivity can introduce serious errors to S_y determination.

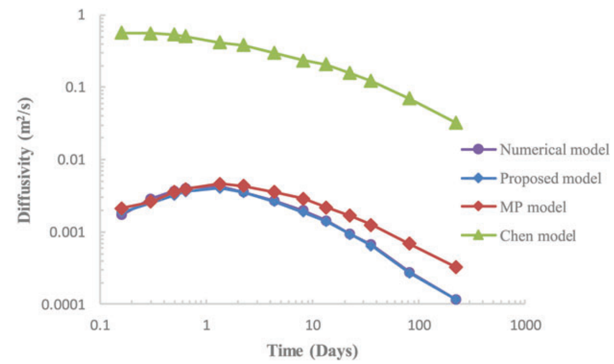

Figure 8. Diffusivity-time curves of the unconfined region by different solutions.

constant during the confined-unconfined conversion. The value is 0.298, almost the same as the true value of 0.3 (Table 1). Assuming that the transmissivity during the confined-unconfined conversion is a constant, the S_y value can be calculated by using the MP model at each time point of interest. As shown in Figure 7, the S_y value calculated using the MP model varies with time and is larger than the true value, suggesting biased estimates due to neglecting the different transmissivities in the unconfined region by the MP model. We note that the approximation of the use of the average spatial diffusivity can result in the incapacity of the Chen model to assess the S_y value of the unconfined region in practice.

Otherwise, the proposed solution can also be used to evaluate the diffusivity of the unconfined region by Equation 18. The results (Figure 8) indicate that the diffusivity of the unconfined region from the proposed solution is identical to that of the numerical solution, suggesting accurate estimates of the diffusivity value. However, the diffusivity-time curves from the MP model and Chen model are obviously deviated from that of the numerical model, indicating spurious assess of the diffusivity value by the use of the conventional analytical solutions. It is suggested that neglecting the difference of hydraulic properties between the confined and the unconfined regions in theory can lead to biased inverse estimates of the dynamic development and the specific yield of the unconfined region by using drawdown data. Hence, the proposed solution is the preferred analytical method for use of analysis of parameters related to the transient confined-unconfined conversion in practice.

Conclusion

A new solution is proposed for drawdown in a confined-unconfined aquifer due to pumping at a constant rate from a fully penetrating well. The flow in the unconfined region is depicted by the Boussinesq equation and the analytical solution is derived by the Boltzmann transform. The proposed analytical solution can be used for drawdown simulation of the transient confined-unconfined flow. Based on a set of drawdown data from pumping test, the proposed analytical solution can also be used to determine the dynamic development of the unconfined region with time and the specific yield of the pumped confined aquifer. Three hypothetical cases are studied to investigate the acceptability and the practical application of the proposed model. The main conclusions are highlighted as follows:

(1) The analytical solution by considering the differences of transmissivity, storativity, and diffusivity during the transient confined-unconfined conversion is firstly derived in the paper. The conventional analytical solutions, namely the MP and the Chen model, are given as special cases of the proposed model.
(2) The difference of the transmissivity, storativity, and diffusivity between the confined and the unconfined regions significantly impacts on the drawdown simulation during the confined-unconfined conversion. The drawdown of the transient confined-unconfined flow can be overevaluated due to the neglect of the variability of transmissivity in the unconfined region and the errors can be accumulative as pumping continues. On the contrary, the drawdown at a given time can be undervalued if the difference of diffusivity is neglected.
(3) The shape of the drawdown-time curve is sensitive to the change of the storativity ratio, S/S_y, between the confined and the unconfined region. The decrease of S/S_y value may diminish and enlarge the slopes of the drawdown-time curves in the confined region and the unconfined region, respectively, at any given time.
(4) The dynamic development of the unconfined region, represented by the radial distance of the conversion interface from the pumping well (R) of the unconfined region can be analyzed by using the proposed solution. The results of comparisons of the proposed solution with the MP model and the Chen model indicate that the use of constant transmissivity of the MP model can lead to overestimations of R values if the flow toward the observation well is under the unconfined condition. The use of the average

spatial diffusivity of the Chen model can result in the biased analysis of the R values during the confined-unconfined conversion.

Hence, the proposed analytical solution provides a comprehensive understanding of the transient confined-unconfined flow induced by a fully penetrating well. It can be used for the design of mine drainage and the management of groundwater resource in the mining area.Acknowledgments

This research was partly supported by the National Natural Science Foundation of China (Grant Numbers 51469002 and 51009029), the Natural Science Foundation of Guangxi (Grant Numbers 2017GXNSFBA198087 and 2014GXNSFBA118263), and the Systematic Project of Guangxi Key Laboratory of Disaster Prevention and Structural Safety (Grant Number 2016ZDX004). We would like to thank two anonymous reviewers for constructive comments and the Editor who help us improve the quality of the manuscript.

Authors' Note

The author(s) does not have any conflicts of interest.

Supporting Information

Additional supporting information may be found online in the Supporting Information section at the end of the article. Supporting Information is generally *not* peer reviewed.

Appendix S1. Derivation processes of the analytical solution of the transient unconfined flow

Appendix S2. Derivation processes of the analytical solution of the transient confined flow

Appendix S3. New derivation of the MP model and the Chen model

Table S1. Drawdown data at the time point of interest in *Case I*

Table S2. Radial distance of conversion interface from the pumping well, R, by different solutions in *Case III*

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