Statistical Dating of the Phenomena of Eudoxus

Dennis Duke, Florida State University

In about 275 B.C. Aratus wrote *Phenomena*, a poem describing, among other things, the arrangement of the constellations relative to each other and relative to the principal circles on the celestial sphere: the equator, the northern and southern tropics, the ecliptic, and the arctic and antarctic circles.[1] We know from the extensive *Commentary* of Hipparchus, *ca.* 130 B.C., that Aratus' main and possibly exclusive source was Eudoxus, who in about 370 B.C. wrote two books, *Phenomena* and *Mirror*, giving essentially the same descriptions that we find in Aratus, plus some additional material – principally lists of constellations on the colures and arctic circles – that we know only through Hipparchus.[2]

We therefore know that Eudoxus had a fully developed conception of the celestial sphere. He understood the importance of the celestial poles and the celestial equator, and that the path of the Sun – the ecliptic – is a circle inclined to the equator. He understood the tropics as the circles parallel to the equator that touch the ecliptic at its most northern and southern points – the solstices, and when the Sun was at a solsticial point, he knew the fraction of the circumference of a tropic above and below the horizon. He understood the colures as circles through the celestial poles and the solsticial points, and through the celestial poles and the equinoctial points, the points where the equator and ecliptic intersect. He understood that the solsticial and equinoctial points are precisely a quadrant apart on both the equator and the ecliptic, and that the two colure circles intersect at right angles at the poles. He understood that stars above the arctic circle are always above the horizon, and hence always visible on any night, and that stars below the antarctic circle are never above the horizon. Eudoxus describes the zodiac as a band bisected by the ecliptic, and he names the sequence of twelve zodiacal constellations that we still use today. Furthermore, he names constellations, and usually specific parts of constellations, that lie on the major celestial circles. Eudoxus is thus the earliest surviving source that describes the fully developed celestial sphere and, what is most important for our considerations, the relation of those circles to the constellations.

We may also infer, with at least some level of confidence, what Eudoxus did not know, or saw no reason to mention. It seems unlikely that he gave values for the height of the celestial north pole above the horizon or for his geographical latitude (which would be the same numbers, of course). Likewise, it is unlikely that he gave any information about the size of the arctic or antarctic circles. It also seems unlikely that he gave a value for the obliquity of the ecliptic, even a round number such as 24° or 1/15th of a full circle. It seems equally unlikely that he gave any actual numbers characterizing the position of constellations or stars relative to the principal celestial circles, or that he imagined any coordinate system of any kind beyond the circles already mentioned. Indeed, it seems unlikely that he gave any direct measures of position whatsoever, since while Hipparchus left us plenty of his own numbers in his *Commentary*, he does not mention any position numbers from Eudoxus.

Instead, what seems to have concerned Eudoxus more than quantitative spatial measurements are temporal relations, both daily and annual. For example, he gives two ratios for the length of longest day to shortest day (5/3 and 12/7). While Hipparchus knew that these can be used to specify geographical latitude, it is not clear that Eudoxus knew that. Eudoxus gives the constellations that rise and set simultaneously with the rising of each zodiacal constellation for the stated purpose of knowing when to expect sunrise. We know from the Geminus parapegma that Eudoxus tabulated the dates of heliacal rising and setting for several bright stars, and he gives the dates in that calendar for autumn equinox and winter solstice.[3] Indeed, it seems most likely that Eudoxus' interest in the tropics and equator was prompted mostly from observation of the annual north-south excursion of the rising and setting points of the Sun on an arc along the eastern and western horizons.

What we may know even less about is when and where Eudoxus' model of the celestial sphere was developed, and who developed it. Presumably some astronomical observations were made that underlie the information in Eudoxus' books, and if we could somehow assign a reliable date, or even a range of dates, to those observations, we would at least know that the celestial sphere was developed no earlier than the observation dates. It is therefore of some interest to use the information from Eudoxus to try and assign dates and possibly locations to the underlying observations.

---

[1] D. Kidd, *Aratus Phenomena*, (Cambridge, 1997).

[2] I am using the English translation of Roger MacFarlane (private communication) with the assistance of Paul Mills. Until this is published, the interested reader must use Hipparchus, *In Arati et Eudoxi phaenomena commentariorium*, ed. and transl. by K. Manitius (Leipzig, 1894), which has an edited Greek text and an accompanying German translation.

[3] G. Aujac, *Geminus Introduction to the Phenomena* (Paris, 1975).

Dating a Star catalog

Before we perform a statistical analysis to date the phenomena of Eudoxus, it will be useful to review a simpler problem: the statistical analysis required to date a star catalog. For simplicity, let's assume we have a catalog listing a set of ecliptic longitudes and latitudes for known stars. Since the longitudes change with time due to precession, we can attempt to date the catalog by comparing the catalog longitudes L_i with the theoretically computed longitudes $\lambda_i(t)$. Assuming that the errors in the catalog longitudes,[4]

$$L_i - \lambda_i(t) = \varepsilon_i$$

are normally distributed with variance σ^2 and mean zero, *i.e.* $N(0,\sigma^2)$, then we can find the best fit time t_{min} by minimizing

$$\chi^2 = \sum_{i=1}^{N} \frac{(L_i - \lambda_i(t))^2}{\sigma^2}$$

Naively, and as we shall see incorrectly, the uncertainty σ_t in the determination of t_{min} can be determined from

$$\chi^2(t_{\min} \pm \sigma_t) - \chi^2(t_{\min}) = 1$$

and is approximately

$$\sigma_t^2 = \frac{\sigma^2}{p^2 N}$$

where p is the precession constant (about 1.4° per century). It is clear that the size of σ_t can be made smaller and smaller by using more stars N.

There is another easy, and equally naïve and incorrect, way to determine the uncertainty σ_t, and that is to use a Monte Carlo simulation. Having determined t_{min} as above, we simply construct a large number of new pseudo-catalogs, perhaps 1,000 or more,

$$\{L_i' = \lambda_i(t_{\min}) + \varepsilon_i',\ i = 1...N\}$$

where the ε_i' are $N(0,\sigma^2)$. Then for each set we minimize χ^2 and determine $t'_{\min}$. The standard deviation of these $t'_{\min}$ values will be an estimate of σ_t.

Two final notes:

(a) All of the above analyses assume that the errors are *uncorrelated*, *i.e.* while $<\varepsilon_i^2> = \sigma^2$, we must also have $<\varepsilon_i \varepsilon_j> = 0$ (where <..> is the usual statistical expectation value).

(b) In the Monte Carlo it is essential that each of the pseudo-catalogs be *possible* sets of observations, even though none is the same as the observed catalog. Another way of saying this is that each set of errors *could* have been the set given by the author, and has the same statistical distribution as the set the author did give.

As it happens, and as discussed below, both of these conditions lead to severe problems when computing the uncertainty in the date naively.

The Effect of Calibration Errors in the Star Catalog

Now let us suppose that the original observer had a calibration error in his measurements, *i.e.* he misplaced his zero-point in longitude with respect to the theoretically correct point that we are using for the λ_i*'s*. In practice, such an error is guaranteed to happen, of course. Thus we would have

[4] In practice, it is necessary to weight the errors by the cosine of the latitude of each star.

$$L_i - \lambda_i(t) = \varepsilon_i + \eta$$

where we can assume that η is $N(0,\sigma_c^2)$, but uncorrelated with the ε_i, so $<\varepsilon_i\eta> = 0$. Now σ_c is the uncertainty in the observer's determination of the zero point in longitude, which for all practical purposes is equivalent to how well he knows the *position of the Sun relative to the stars* on the day of some cardinal event, *i.e.* an equinox or solstice. Since σ^2 is the variance in the positioning of the stars themselves, then it seems the most reasonable assumption is that σ_c should be at least as big as σ. In any event, it is clear that in the presence of a calibration error the errors in $L_i - \lambda_i(t)$ are *correlated*, and the analyses outlined above must be done differently.

We first compute the covariance matrix of the errors. The diagonal terms in this matrix are just

$$V_{ii} = <(\varepsilon_i + \eta)(\varepsilon_i + \eta)> = \sigma^2 + \sigma_c^2$$

while the off-diagonal terms, which are of course zero in the case of uncorrelated errors, are

$$V_{ij} = <(\varepsilon_i + \eta)(\varepsilon_j + \eta)> = \sigma_c^2$$

Now we have to minimize

$$\chi^2 = \sum_{i=1}^{N} (L_i - \lambda_i(t)) V_{ij}^{-1} (L_j - \lambda_j(t))$$

which clearly reduces to the familiar case when V is diagonal. The uncertainty σ_t in the determination of t_{min} can still be estimated from

$$\chi^2(t_{\min} \pm \sigma_t) - \chi^2(t_{\min}) = 1$$

and is now approximately

$$\sigma_t^2 = \frac{\sigma^2}{p^2 N} + \frac{\sigma_c^2}{p^2}$$

Contrary to what we found assuming uncorrelated errors, it is clear that now σ_t cannot be smaller than σ_c/p, and so can *no longer* be made arbitrarily small by using more stars N. In fact, if σ_c is about the same size as σ, then the final uncertainty in $p\sigma_t$ cannot be smaller than the uncertainty in the longitude of a *single* star.

Dating the Phenomena of Eudoxus

Now let us now suppose that instead of a star catalog, which of course gives in one way or another the author's determination of the positions of a collection of stars, we have the statements of Eudoxus, which have come down to us in two ways: first, through the collection of what are apparently direct quotes from Eudoxus by Hipparchus in his *Commentary to Aratus and Eudoxus*, and second, through the poem of Aratus, which according to Hipparchus is a fairly accurate paraphrase of Eudoxus' works. The relevant quotations from Eudoxus are given in the Appendix.

Instead of fitting ecliptic longitudes, it is just as easy to fit right ascensions and declinations, especially if we are using the first analysis outlined above, which treats the phenomena as if they are from a star catalog and ignores calibration errors. Indeed, one finds

for the colure data	1150 ± 130 B.C.
for the equator data	950 ± 230 B.C.
for the northern tropic data	1120 ± 280 B.C.
for the southern tropic data	1170 ± 300 B.C.
for all declination data	1070 ± 160 B.C.
for all the data	1130 ± 90 B.C.

in good agreement with previous results.[5] Note that the date 1130 B.C. is about 8σ away from Eudoxus' date 370 B.C.

[5] B. E. Schaefer, "The Latitude and Epoch for the Origin of the astronomical Lore of Eudoxus", *Journal for the History of Astronomy,* xxxv (2004) 161– 223, and references therein.

It is the case, however, that the phenomena of Eudoxus should not be analyzed as if they come from a star catalog. In order to see this, let us initially consider the data for the solsticial colure. This is a line of constant ecliptic longitude (as well as constant right ascension). Thus the ‘measured values’ L_i are all either 90° or 270°, and we compute the various $\lambda_i(t)$ as before. For the moment let us ignore the calibration error, and consider just the statistical errors

$$L_i - \lambda_i(t) = \varepsilon_i$$

We might minimize χ^2 and determine t_{min} using

$$\chi^2 = \sum_{i=1}^{N} \frac{(L_i - \lambda_i(t))^2}{\sigma^2}$$

and proceed as described above, determining σ_t from $\chi^2(t_{\min} \pm \sigma_t) - \chi^2(t_{\min}) = 1$, and adjusting σ in each fit so that the χ^2 per degree of freedom is about unity.

On the other hand, we might try instead to use a Monte Carlo simulation to determine σ_t.
The first task is to generate a new set of errors

$$\{L_i' = \lambda_i(t_{\min}) + \varepsilon_i',\ i = 1 \ldots N\}$$

Suppose that the first star is supposed to be on the 90° colure. We can generate the first error, ε_1', but then we must have that

$$90^\circ - \lambda_1(t_{\min}) = \varepsilon_1'$$

Note, however, that once we know the error in the position of the first star with respect to the supposed solsticial colure, *we automatically know the value of t*: it is the time t_1' when the true colure at time t_1' is a distance (in longitude) ε_1' from the position of the star at time t_{min}. Suppose we try to generate the error in position for the second star, also said by Eudoxus to be on the 90° colure. Picking a random ε from $N(0,\sigma^2)$, we would get

$$90^\circ - \lambda_2(t_{\min}) = \varepsilon_2'$$

and from this a t_2' as before. However, it is in general statistically *impossible* that $t_1' = t_2'$, and so this set of errors is not a physically realizable set, and cannot be used in a valid Monte Carlo simulation.

In fact, it is easy to see in this case that once we know the error in the first star, which is after all just the distance from a specified circle of constant longitude, we can use the theoretically known position of the second, and all subsequent, stars to *compute* their positions relative to *that same circle*, and hence their errors *must* all be computed and not generated as random variables.

It is furthermore clear that this analysis generalizes to circles of constant right ascension and declination. Once we know the distance of the first star from such a circle, we can find the unique time t_1' when that error would be realized, and we must use exactly that same time to compute the positions of all the other stars, and hence their distances from any circle. To do otherwise would be to create a set of errors and star positions that is not physically possible, and would not be acceptable in a Monte Carlo simulation.

In practice, this all means that the uncertainty in the determination of the time is proportional to the uncertainty in position of a *single* star, so

$$\sigma_t^2 = \frac{\sigma^2}{p^2}$$

In the presence of a calibration error (which, of course, should not be ignored in any event) we then get

$$\sigma_t^2 = \frac{\sigma^2 + \sigma_c^2}{p^2}$$

Thus if σ_c is about the same size as σ, then the total uncertainty in $p\sigma_t$ is about $\sqrt{2}\,\sigma$. For $\sigma = 5°$, which is the correct average value for all the data, the uncertainty in t is then about $\sigma_t = 500$ y, and so the difference in the dates 1130 B.C. and 370 B.C. is about an 8σ effect when computed naively, but only a 1.5σ effect when computed correctly.

There is another consideration that should be taken into account in any statistical analysis of historical data that span many centuries. One justification for minimum χ^2 analyses comes from consideration of *likelihood*. In this case, there is only one model parameter to be determined, the time t, and the likelihood assumption is that at any time t, if the errors $\varepsilon_i = L_i - \lambda_i(t)$ are independently distributed according to some probability density $f(\varepsilon_i;t)$, then the likelihood of observing the values $\varepsilon = \{\varepsilon_i, i = 1,...,N\}$ at that time t is

$$L(\varepsilon \mid t) = \prod_{i=1}^{N} f(\varepsilon_i;t),$$

and the most likely value of t is that which maximizes the likelihood. If $f(\varepsilon_i;t)$ is the normal distribution,

$$f(\varepsilon_i;t) = \frac{1}{\sigma_i\sqrt{2\pi}} e^{-\frac{\varepsilon_i^2}{2\sigma_i^2}},$$

then χ^2 is, up to irrelevant constants, just $-2\ln L(\varepsilon \mid t)$, so clearly the maximum likelihood occurs at the same value of t that minimizes χ^2.

More generally, however, we can ask what is the probability density $p(t \mid \varepsilon)$ of t given the observed error set $\varepsilon = \{\varepsilon_i, i = 1,...,N\}$. That is given by Bayes' theorem as

$$p(t \mid \varepsilon) = \frac{L(\varepsilon \mid t)\pi(t)}{\int L(\varepsilon \mid t')\pi(t')dt'}$$

where $L(\varepsilon \mid t)$ is the likelihood defined above, and $\pi(t)$ is the prior probability density of t, reflecting whatever knowledge we might have of t before we consider the data set ε.[6]

Note that if we have no prior information on t then $\pi(t)$ is simply a constant, corresponding to a uniform probability distribution, and Bayes' theorem reduces to conventional maximum likelihood or minimum χ^2. In practice, of course, we must obviously admit that $\pi(t)$ can be constant only over some appropriate interval of time, so, for example, no one would accept an analysis indicating that Eudoxus' data was measured in, say, 4000 B.C. or A.D. 2000. What we learn from this is that if the only information we have about t in the case of Eudoxus' on-circle data is the data itself, then it is correct to perform a conventional minimum χ^2 analysis as we did above. But if we have any other independent information on t, then we should include it according to Bayes' theorem.

In fact, one thing we do know is that Eudoxus' lore is given in the context of a fully developed model of the celestial sphere. Thus, when Eudoxus says that a constellation is on a colure, he obviously must know what a colure is, and where it is located with respect to the visible stars. If we conclude that the lore date to a time much earlier than Eudoxus, then it must be the case that the celestial sphere was fully developed at that earlier time. So if we say that $\pi(t)$ is a constant over some interval of time, one of the things we are implicitly saying is that the knowledge of the celestial sphere and its associated cosmology was constant over that same interval. For the specific case of the on-circle data, we would be saying that *the model of the celestial sphere known to Eudoxus in, say, 370 B.C., was equally well known in, say, 1120 B.C.*

Now I strongly doubt that many historians would agree with that last statement. In disputing it, they would point out that, setting aside the on-circle data of Eudoxus, no evidence has come down to us suggesting that *any* culture prior to

[6] this is all very standard material that is discussed in many places. See, e.g., Glen Cowan, *Statistical Data Analysis*, (Oxford, 1998) 93-94.

Eudoxus' time understood the cosmology of the celestial sphere. For example, the stories from Homer and Hesiod mention a few astronomical facts, but nothing approaching the celestial sphere. The same is true for all known Greek sources earlier than Eudoxus.[7] The cuneiform texts from Babylon and Uruk give no hint that the Babylonians understood the celestial sphere. Furthermore, the information in MUL.APIN, which is similar in some respects to the phenomena of Eudoxus and which probably dates to late second millennium B.C., gives no hint of a celestial sphere.[8]

Considering all of these cases together, it strains credulity to the breaking point that each and every source we know from the time before Eudoxus might have known about the celestial sphere in all its details, but either chose not to write anything about it, or if they did, it has not reached us, even indirectly through intermediate sources such as Hipparchus. Therefore, it appears that if we invoke the knowledge we have independent of Eudoxus, we might tolerate a uniform prior for perhaps a century before Eudoxus, but we should most certainly not be assuming a uniform prior for the millennium or more predating Eudoxus.

The implementation of the prior knowledge of t in a statistical analysis is unavoidably somewhat subjective, but we can easily imagine reasonable approaches to the issue. One simple way to proceed is to assume that the celestial sphere was developed no later than some time t_0 and over some time interval τ, and use a function that approaches zero for times earlier than t_0, with the rate of approach controlled by the time interval τ, and is strictly zero for times later than t_0. One such probability distribution is the truncated normal,

$$\begin{aligned}\pi(t) &\simeq \exp\frac{-(t-t_0)^2}{2\tau^2}, \quad t \le t_0 \\ &= 0, \quad t > t_0\end{aligned}$$

For the case of the cosmology of the celestial sphere, the choices $t_0 = 400$ B.C. and $\tau = 50$ yrs might be appropriate, although there is no way to know these parameters with any certainty. Adding to χ^2 the term $-2\ln\pi(t)$ and minimizing the resulting sum leads to an estimated date of 550 ± 50 B.C. Clearly, this estimate is effectively determined entirely by the assumed values of t_0 and τ, and so there is some truth in saying that we have essentially assumed the answer. In reality, though, what has been done is to simply enforce in the statistical analysis the very strong belief, founded on substantial historical data, that the celestial sphere did not precede Eudoxus by very long. Similarly, to *omit* the prior is to effectively assert an equally strong belief that invention of the concept of the celestial sphere is uniformly likely at any time over the millennium preceding 370 B.C.

Of course, a more direct and for all practical purposes equivalent strategy is to go ahead and perform a standard χ^2 analysis ignoring the prior information, and if the result is found to conflict with the prior information, we simply discard the result as unreliable and say instead that Eudoxus, or some near contemporary, made errors sufficiently large to account for the observations he used.[9]

Altogether, we must conclude that statistical analysis provides no significant evidence that the phenomena of Eudoxus were not originated sometime near to 370 B.C.

Appendix

Hipparchus' quotations from Eudoxus list which constellations are on the colures, the equator, and the two tropics (omitting for now the two arctic circles).[10] For the equator and tropics Aratus gives similar lists, but whenever we have the direct quotations from Eudoxus, there is no reason to use the information from Aratus. There are several cases where it appears that Aratus is correcting Eudoxus, but we certainly do not want to use that information when attempting to date the observations underlying Eudoxus' statements.

The Solsticial Colure

---

[7] See, e.g. David Dicks, *Early Greek Astronomy to Aristotle*, (Ithaca, 1970).

[8] Herman Hunger and David Pingree, *Astral Sciences in Mesopotamia*, (Leiden, 1999); James Evans, *The History and Practice of Ancient Astronomy*, (Oxford, 1998) 5-8.

[9] A. Gelman *et al.*, *Bayesian Data Analysis*, (Boca Raton, 2000) 259-262. It is perhaps interesting that this was exactly the conclusion of the discussion by Dicks (see ref. 7, p 162-3 and p 250, n 265) of R. Böker, 'Die Enstehung der Sternsphaere Arats', *Berichte über die Verhandlungen der sächsischen Akademie der Wissenschaften zu Leipzig,* **99** (1952) 3-68. Böker's statistical analysis puts the epoch of Eudoxus' phenomena as 1000 ± 30–40 B.C. and the latitude of the observer as between 32°30´ and 33°40´.

[10] The following quotations are all from the translation of Roger MacFarlane (ref. 2).

> 1.11.9 Further, Eudoxus treats also the stars which lie upon the so-called colures, and says that the Great Bear's middle lies upon one of them, and also the Crab's middle, the Water-snake's neck, and the part of Argo between the prow and the mast; then it is drawn after the invisible pole through the tail of the Southern Fish, the Capricorn's middle, and the middle of Arrow; then through the Bird's neck, its right wing, and through Cepheus' left hand; and through the bend of the Snake and beside the Small Bear's tail.

Solsticial Colure	DD	PK
the Great Bear's middle	bet Uma	25
the Crab's middle	M44	449
the Water-snake's neck	the Hya	900
the part of Argo between the prow and the mast		880
the tail of the Southern Fish	gam Gru	1022
the Capricorn's middle	eta Cap	618
the middle of Arrow	del Sge	283
the Bird's neck	eta Cyg	161
the Bird's right wing	kap Cyg	167
Cepheus' left hand	the Cep	80
the bend of the Snake	chi Dra	61
beside the Small Bear's tail	zet Umi	4

The Equinoctial Colure

> 1.11.17 In the other colure, he says that there lie first the left hand of Arctophylax and his middle, taken lengthwise; then the middle of the Claws, taken breadth-wise, and the right hand of the Centaur and his front knees; then after the invisible pole the bend of the River and the Sea-monster's head and the back of the Ram, taken breadth-wise, and the head of Perseus and his right hand.

Equinoctial Colure	DD	PK
the left hand of Arctophylax	kap Boo	88
the middle of Arctophylax, taken lengthwise	alp Boo	110
the middle of the Claws, taken breadth-wise	alp Lib	529
the right hand of the Centaur	kap Cen	951
the front knee's of the Centaur	alp Cen	969
the bend of the River	rho Eri	786
the Sea-monster's head	gam Cet	714
the back of the Ram, taken breadth-wise	alp Ari	375
the head of Perseus	the Per	194
the right hand of Perseus	CG869	191

The Equator

For most of the constellations, Hipparchus does not quote explicitly from Eudoxus, but from Aratus, who wrote [511-524]:

> As a guide the Ram and the knees of the Bull lie on it, the Ram as drawn lengthwise along the circle, but of the Bull only the widely visible bend of the legs. On it is the belt of the radiant Orion and the coil of the blazing Hydra, on it too are the faint Bowl, on it the Raven, on it the not very numerous stars of the Claws, and on it the

> knees of Ophiuchus ride. It is certainly not bereft of the Eagle: it has the great messenger of Zeus flying near by; and along it the Horse's head and neck move round.

Hipparchus then adds that Eudoxus gives the flowing additional information:

> 1.10.22 Eudoxus expressed the rest similarly; but, he says that the middle of the Claws lies on the equator, and that the left wing of the Eagle, the rump of the Horse, and also the northern of the Fishes do also.

Equator	DD	PK
the Ram as drawn lengthwise	alp Ari	375
the Bull, only the widely visible bend of the legs	mu Tau	386
the belt of Orion	eps Ori	760
the coil of Hydra	alp Hya	905
the Bowl	del Crt	923
the Raven	gam Crv	931
the Claws	alp Lib	529
the knees of Ophiuchus	zet Oph	252
the left wing of the Eagle	alp Aql	288
the rump of the Horse	gam Peg	316
the Horse's head (Aratus)	eps Peg	331
the Horse's neck (Aratus)	zet Peg	325
the northern of the Fishes	eta Psc	695

The Summer Tropic

> 1.2.18 Concerning the stars which are borne upon the summer and winter tropics, and also upon the equator, Eudoxus says this about the summer tropic:
> Upon it are: the middle of the Crab, the parts lengthwise through the middle of the Lion, the area a little above the Maiden, the neck of the held Snake, Engonasin's right hand, Ophiuchus' head, the Bird's neck and its left wing, the Horse's feet, but also Andromeda's right hand and the part between her feet, Perseus' left shoulder and left shin, and also the Charioteer's knees and the Twins' heads. It then concludes near the middle of the Crab.

Summer Tropic	DD	PK
the middle of the Crab	M44	449
lengthwise through the middle of the Lion	eta Leo	468
the area a little above the Maiden	eps Vir	509
the neck of the held Snake	del Ser	269
Engonasin's right hand	kap Her	122
Ophiuchus' head	alp Oph	234
the Bird's neck	eta Cyg	161
the Bird's left wing	eps Cyg	168
the Horse's feet	pi Peg	332
Andromeda's right hand	rho And	340
the part between her [Andromeda's] feet	gam And	349
Perseus' left shoulder	the Per	194
Perseus' left shin	xi Per	214

the Charioteer's knees	chi Aur	231
the Twins' heads	bet Gem	425

The Winter Tropic

> 1.2.20 About the winter tropic, Eudoxus says this:
> Upon it are: the middle of the Capricorn, the feet of the Water-pourer, the Sea-monster's tail, the River's Bend, the Hare, the Dog's feet and tail, the Argo's prow and mast, the Centaur's back and chest, the Beast, and the Scorpion's stinger. Then proceeding through the Archer it concludes at the middle of the Capricorn.

Winter Tropic	DD	PK
the middle of the Capricorn	eta Cap	618
he feet of the Water-pourer	del Aqr	646
the Sea-monster's tail	iot Cet	732
the River's Bend	rho Eri	786
the Hare	alp Lep	812
the Dog's feet	zet Cma	834
the Dog's tail	eta Cma	835
the Argo's prow		879
the Argo's mast	alp Pyx	876
the Centaur's back	nu Cen	946
the Centaur's chest		
the Beast	del Lup	974
the Scorpion's stinger	lam Sco	565
the bow of the Archer (Aratus)	del Sgr	571

Hipparchus also writes:

> 2.1.20 Eudoxus makes it clear in the following statement that he places the tropic points at the middles of the zodiacal signs: "There is a second circle [the northern tropic], on which the summer solstices occur; and on this is the middle (parts) of the Crab." Again he says, "There is a third circle [the equator] on which the equinoxes occur; and on this are both the middle (parts) of the Ram and the Claws. And there is a fourth [the southern tropic] on which the winter solstices occur; on it is the middle (parts) of the Capricorn." He states it yet more conspicuously in the following, for the so-called colures, which are drawn through the poles and the solstices and the equinoxes, he says: "There are two other circles through the poles of the cosmos, which cut one another in half, and at right angles. The constellations upon these lines are the following: first the ever-visible pole of the cosmos, then the middle of the Bear, reckoned breadth-wise, and the middle of the Crab." Then a little later he says, "Both the tail of the Southern Fish and the middle of the Capricorn." In later passages he says that in the other of the circles through the poles lie among others, which he enumerates, the middle (parts) of the Claws, reckoned breadth-wise, and the back (parts) of the Ram, reckoned breadth-wise.

The precise meaning of these passages is connected with how Eudoxus is treating the relationship of the zodiacal constellations and the zodiacal signs.[11] Hipparchus clearly thinks that when Eudoxus says 'the middle (parts) of the Claws" he is referring to the middle of the zodiacal sign. We know this because Hipparchus repeatedly tells us that Eudoxus has arranged his signs so that the solstices and equinoxes occur at the middle of the signs.

[11] The question of just what Eudoxus meant, as opposed to what Hipparchus says he meant, is dealt with in detail by Bowen, A.C., & Goldstein, B.R. "Hipparchus' Treatment of Early Greek Astronomy: The case of Eudoxus and the length of daytime". *Proceedings of the American Philosophical Society* 1991, 135: 233-254.

However, as discussed in detail by Bowen and Goldstein, it is not at all clear that Eudoxus was, as Hipparchus thought, referring to the signs of the zodiac when he mentions the middle of the Crab, the Claws, the Ram, and the Capricorn. Instead, by analogy with his coincident statements that the colure also goes through, e.g. the Great Bear's middle, Eudoxus might well have been referring not to the signs but to the constellations. Of course, it is also possible that Eudoxus was considering the zodiacal signs and constellations as equivalent in some sense.

In addition, it is also not at all clear that Eudoxus was referring to anything as specific as the *midpoint* of either the sign or the constellation, since his use of the plural (τά μέσα) implies simply 'in the interior', and nothing as specific as a central position.

Therefore, for our purposes we can safely assume that Eudoxus understood that

(a) the colures are great circles that intersect at right angles at the north celestial pole, so that one colure goes through the two solsticial points, the other through the two equinoctial points;
(b) the ecliptic and the tropics touch at the solstices, the ecliptic and the equator cross at the equinoxes, and neighboring cardinal points are exactly one quadrant apart on both the equator and the ecliptic;
(c) the solsticial colure goes through the middle parts of the Crab and the Capricorn, while the equinoctial colure goes through the middle parts of the Ram and the Claws
(d) the solsticial and equinoctial points mark a location where Eudoxus thought the Sun was on a particular day of the year, but he specifies the location no more precisely than the middle parts of the Crab on summer solstice, the middle parts of the Claws on autumn equinox, etc.
(e) various other specified constellations and constellation parts lie on or nearby the circles which define the celestial sphere.

So while we may be relatively sure that Eudoxus knew the *date* of, say, summer solstice, to an accuracy of a few days, perhaps by looking for the turnings of the Sun on the eastern and western horizons in summer and winter, we have no specific information about how he might have determined the sidereal location of the Sun on those dates, or on the equinoxes. One plausible explanation is that he observed the date of summer solstice by observing the most northerly setting of the Sun on the western horizon, and then observed which constellations rose at sunrise on nearby mornings.