Finite Element Treatment of the Navier Stokes Equations:
Part I: The Navier Stokes Equations

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15 August 2021

1 Introduction

These notes have the goal of explaining how the finite element method was used to solve a class of flow problems governed by the Navier Stokes equations.

In this part, we consider a fairly general version of the Navier Stokes equations, but we will explain the steps involved in simplifying those equations so that they are appropriate for the problems we actually wish to consider, namely steady state, incompressible flow in a two dimensional region.

Once we have the mathematical equations defined, we will consider how these equations might be solved. We show how the system can be regarded as a system of nonlinear equations, and what the form of Newton's method would be. Since our unknowns are not scalars or vectors, but arbitrary functions, our discussion of Newton's method is merely formal at this point.

In the next part, we will show how ideas from the finite element method can be used to develop a way of approximating the state functions by discretely describable functions. These functions will be formed from simple functions based on a discretization of the problem geometry. Once we have a discretized representation of velocity and pressure fields, the Galerkin procedure will allow us to set up a system of equations for the coefficients of the basis functions, which are satisfied by the approximate solution of the discretized Navier Stokes equations.

Later, we will turn to the text of a computer program that embodies these ideas. Going through the text of that program, we will explain how Newton's method, the Galerkin procedure, the finite element approximation, and the Navier Stokes equations all come together.

2 The Time-Dependent Compressible Navier Stokes Equations

A fluid flowing in a region has the obvious physical properties of a vector velocity $\mathbf{v}(\mathbf{x}, t)$ and a scalar density $\rho(\mathbf{x}, t)$. A third physical property, somewhat more difficult to measure and understand, is the *fluid pressure*, a scalar symbolized by $p(\mathbf{x}, t)$.

For convenience, we may consider the product of velocity and density, called *the mass velocity*, and symbolized by $\mathbf{u}$:

$$\mathbf{u}(\mathbf{x}, t) = \rho(\mathbf{x}, t) \cdot \mathbf{v}(\mathbf{x}, t) \tag{1}$$

The behavior of pressure, density, and velocity is modeled by the *time-dependent compressible Navier-Stokes equations*:

$$\rho \mathbf{v}_t - \mu \Delta \mathbf{v} + \rho (\mathbf{v} \cdot \nabla) \mathbf{v} + \nabla p = 0 \tag{2}$$
$$\rho_t + \rho \nabla \cdot \mathbf{v} = 0 \tag{3}$$

The first equation is known as the *momentum equation*, the second as the *continuity equation*.

3 The Fluid Viscosity

In the Navier Stokes equations, the quantity μ is the value of a physical property known as the *dynamic viscosity*. In the cases we will examine, the dynamic viscosity is treated as a constant. However, it is not unusual to consider situations in which the viscosity may depend on some other parameter, such as temperature. The value of μ greatly affects the properties of the flow, and the difficulty of obtaining approximate solutions to the flow.

Roughly speaking, high values of μ mean the fluid is "sticky" or viscous, like honey. Flows of such fluids tend to be orderly and simple. Low viscosity fluids, on the other hand, are much more susceptible to disorderly, turbulent flows. One reason for this is that μ measures the tendency of differences in the flow to be smeared out. In honey, if one particle tries to move very quickly, this motion is quickly replaced by a smoothed out, slower motion of all the particles in a neighborhood.

4   The Steady State Compressible Navier Stokes Equations

One simplification to the Navier Stokes equations occurs if we can assume that the flow problem has a steady behavior. For instance, if we turn on the tap in a sink, the flow will at first be very irregular; however, once the sink has filled up, and the water is pouring out over the sides as fast as it is pouring in, at least the rough form of the flow will be the same over time.

   We might imagine that, to a reasonable approximation, even the small scale structure of this flow becomes fixed as time progresses. A flow which does not depend on time is known as a *steady state flow.*

   In cases where we know, or hope, that the fluid flow problem has a steady state solution, we can certainly try to determine that solution by solving the Navier Stokes equations and looking at the behavior of the solution as time increases indefinitely.

   An alternative way to reach the steady state solution is to assume that the variables $\mathbf{v}$, ρ, and p do not depend on time, plug them into the Navier Stokes equations, and after simplification, arrive at the *steady state compressible Navier Stokes equations*:

$$-\mu\Delta\mathbf{v} + \rho(\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla p \;\;=\;\; 0 \tag{4}$$

$$\rho\nabla\cdot\mathbf{v} \;\;=\;\; 0 \tag{5}$$

5   The Steady State Incompressible Navier Stokes Equations

Any physical fluid or gas will have variations in density. In many cases, especially when (temperate) water is involved, these variations are so small that the density may be taken to be a constant, as though the fluid were incompressible. In that case, we may simplify the equations by dividing through by the density to get the *steady state incompressible Navier Stokes equations*:

$$-\frac{1}{\rho}\Delta\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \frac{1}{\rho}\nabla p \;\;=\;\; 0 \tag{6}$$

$$\nabla\cdot\mathbf{v} \;\;=\;\; 0 \tag{7}$$

   To make our equations slightly less cluttered, we replace the dynamic viscosity by the *kinematic viscosity* ν:

$$\nu = \frac{\mu}{\rho} \tag{8}$$

and we replace the pressure by a scaled pressure, $\bar{p}$:

$$\bar{p}(\mathbf{x}) = \frac{p(\mathbf{x})}{\rho} \tag{9}$$

   However, pressure is not an "interesting" quantity, so we often don't care about its particular values or scaling. For convenience, then, we will ignore this rescaling of the pressure, and persist in using the old symbol p when in fact we will mean the new, rescaled pressure. At this point, our cleaned up version of the incompressible steady Navier Stokes equations looks like this:

$$-\nu\Delta\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla p \;\;=\;\; 0 \tag{10}$$

$$\nabla\cdot\mathbf{v} \;\;=\;\; 0 \tag{11}$$

   By using the appropriate scalings, our set of equations for the kinematic velocity v look as simple as the previous set of equations for the mass velocity u, but are in fact simpler, since we have eliminated consideration of the density ρ.

   This is the vector form of the Navier Stokes equations that we will be considering from now on. Our only further modification is to specify the spatial dimension, and, for convenience, to write out a scalar version of the equations.

6 The Steady Incompressible Navier Stokes Equations in Two Space Dimensions

It might be helpful now to assume that we are working in a two-dimensional spatial region, and to write the steady incompressible Navier Stokes equations in a sort of scalar form. Thus, instead of $\mathbf{x}$ we will write (x, y), and instead of the vector quantity $\mathbf{v}$ we will write (u, v), where we are keeping track of the horizontal and vertical components of the kinematic velocity.

From this perspective, our system now becomes three scalar equations, since our vector momentum equation is replaced by a pair of horizontal and vertical momentum equations:

$$-\nu(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}) + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{\partial p}{\partial x} = 0 \tag{12}$$

$$-\nu(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}) + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \frac{\partial p}{\partial y} = 0 \tag{13}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{14}$$

7 The Reynolds Number

We mentioned earlier that the dynamic viscosity μ influenced whether a fluid was likely to flow in a motion that was orderly and diffusion-dominated, corresponding to high values of μ) or a turbulent momentum-dominated motion, for low values of μ.

Since ν is simply a multiple of μ, it is also a measure of this same tendency.

For any fixed value of ν, we can make the flow become turbulent if we make it move fast enough. The simplest way for this to occur would be if we were able to specify high velocity values along the boundary. The momentum equations would guarantee that high velocities would try to spread into the region, and the viscosity would control how smoothly or turbulently this happened.

It turns out that a better measure of whether a flow is actually going to be smooth or turbulent requires knowledge of both the dynamic viscosity μ and the magnitude of a "typical" velocity V_0, (say, the maximum). And in order to get a dimensionless number, we also want the values of a "typical" length in the flow region, L_0, and a "typical" value of the density, ρ_0 (which we can take to be the constant value we have already assumed.)

Using this data, we can define a ratio R known as the *Reynolds number*. R can be written in terms of the kinematic viscosity ν_0 if we combine the density and dynamic viscosity μ_0:

$$R \equiv \frac{\rho_0 ||V_0|| L_0}{\mu_0} = \frac{||V_0|| L_0}{\nu_0} \tag{15}$$

The Reynolds number summarizes in one value the competing effects of the scales of length, velocity, mass and viscosity. In the Navier Stokes equations, R controls the relative weight of the linear and nonlinear portions of the momentum equations; as R increases, the nonlinearity grows.

The Reynolds number allows us to make statements about a flow that are independent of the particular scale of the problem. In particular, it says that if we normalize all lengths, velocities, and densities in our problem by dividing them by L_0, $||V_0||$, and ρ_0 respectively, then our flow will be the same as any other flow with the same normalized values.

It also means that, roughly speaking, the value of R tells us when turbulent patterns will dominate in the flow. In engineering experience, a flow with R of the order of 1 through 1000 is generally smooth, with turbulence appearing and strengthening as R increases into the range of 100,000 and more.

It is often convenient to consider the Reynolds number as the important parameter in a flow calculation, rather than the viscosity. We think of the fluid viscosity as a constant, but the Reynolds number is a ratio of quantities, some of which might change. In particular, we might imagine studying a series of flows in

which the maximum velocity increases by a factor of 10 each time. Rather than doing that, we could simply solve the problem repeatedly, multiplying R by 10 each time. This implicitly rescales the problem, without making us change anything but R.

For this reason, it can be desirable to rewrite the Navier Stokes equations so that the Reynolds number appears in front of the nonlinear terms. That means we are dividing through by ν, and assuming that our values of $||V_0||$ and L_0 are about 1. In that case, our vector equations look like this:

$$-\Delta \mathbf{v} + R(\mathbf{v} \cdot \nabla)\mathbf{v} + \nabla p = 0 \tag{16}$$
$$\nabla \cdot \mathbf{v} = 0 \tag{17}$$

We've had to fudge a bit to get the R to show up here. To be honest, we should write $\frac{R}{||V_0||L_0}$ instead. But we usually don't worry about this. If our typical velocity and length aren't close to 1, then little harm is done, except that we have to adjust the value of R when we report our results or make any comparisons with another flow based on Reynolds numbers.

8   A Formal version of the Navier Stokes Equations

We are now ready to *start* considering how to solve the Navier Stokes equations. We begin by writing the equations of state as a single formal equation:

$$\mathbf{F}(\mathbf{v}, p) = \mathbf{f}(\mathbf{x}). \tag{18}$$

Here, $\mathbf{f}(\mathbf{x})$ represents any source terms, which will be taken to be zero throughout the region unless specifically noted.

Normally, a particular flow problem is studied over a limited or specified spatial domain, symbolized by Ω, and no information about the behavior of the flow is assumed or sought outside of Ω. However, the "outside world" exerts its influence on the flow along the boundary $\partial\Omega$. This influence may involve requirements along portions of the boundary that the flow be zero along this boundary, or have a particular normal or tangential component. There is also usually a specification of the pressure at a single boundary point. This combination of requirements is termed the *boundary conditions*; for the moment, we will represent whatever boundary conditions apply to our problem as:

$$\mathbf{G}(\mathbf{v}, p) = \mathbf{g}(\mathbf{x}). \tag{19}$$

We represent the boundary conditions as a single equation that applies at each point $\mathbf{x} \in \partial\Omega$. In practice, of course, the boundary conditions are likely to be expressed as a collection of equations, only some of which apply at any particular boundary point. We will ignore such complications.

9   Newton's Method to the Formal Navier Stokes Equations

You are probably used to seeing Newton's method applied to simple problems involving a few nonlinear equations, in which the unknowns are numerical values. However, it is possible, at least formally, to define a Newton method for our case, in which our unknowns are functions. We will sketch this here, just as a preparation.

Eventually, we will discretize the system. At that point, we will see both the relationship with the formal Newton process defined here in terms of functions, and the common use of Newton's method for systems of (finitely many) algebraic equations.

To begin, we imagine that we have combined our state variables into a single function $\mathbf{w}(\mathbf{x})$, and our equations and boundary conditions into a single formal equation:

$$\mathbf{H}(\mathbf{w}) = 0. \tag{20}$$

To apply Newton's method, it is only necessary that we be able to come up with a linear operator Hp which estimates the value of the function for a given small displacement $d\mathbf{w}$.

What we are saying here is that, if $\mathbf{w}(\mathbf{x})$ is a given set of state functions, we would like to estimate the change in the value of the Navier Stokes equations at a slightly perturbed set of functions $(\mathbf{w} + d\mathbf{w})(\mathbf{x})$.

$$\mathbf{H}(\mathbf{w} + d\mathbf{w}) \approx \mathbf{H}(\mathbf{w}) + \mathbf{Hp}(\mathbf{w}) \cdot \mathbf{dw} \tag{21}$$

If we can define such a linear operator, and if that linear operator is invertible, then Newton's method will follow immediately:

$$\mathbf{w}^{k+1} = \mathbf{w}^k - \mathbf{Hp}^{-1}(\mathbf{w}) \cdot \mathbf{H}(\mathbf{w}^k) \tag{22}$$

This is as far as we can get with the formal equation. In order to proceed, we must now consider how we can develop a discretized space, which is "fine enough" that we can approximate our state functions, but "discrete enough" that we can set up and solve problems computationally.