FreeFem++, part I

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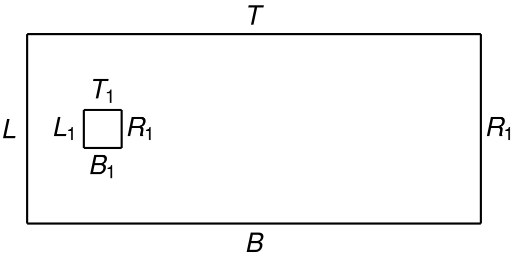
Topics

Topics

Incompressible NSE

$$\frac{\partial u}{\partial t} + u \cdot \nabla u - \nu \Delta u + \nabla p = f$$
$$\nabla \cdot u = 0$$



example18.edp Vortex shedding past a square

example18.edp

`example18.edp` Code outline

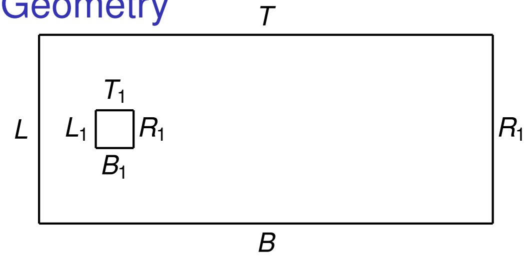
1. Generate/plot geometry
2. Generate/plot mesh
3. Specify finite element spaces and instantiate variables
4. Construct weak form with b.c.
5. Time loop
   5.1 Update time-dependent b.c.
   5.2 Solve
   5.3 Plot

Some syntax (C++)

- Comments preceeded by `//`
  - Can also use `/* ... */`
  - Good for multiline comments
- Commands mostly end with semicolon
  - No need for a line continuation character
- Extra spaces and indentation don't matter
- Variables must have a specified type
- "Curly brackets" (`{ }`) used for grouping
- Single quotes for characters (`'a'`)
- Double quotes for strings (`"Hello there"`)

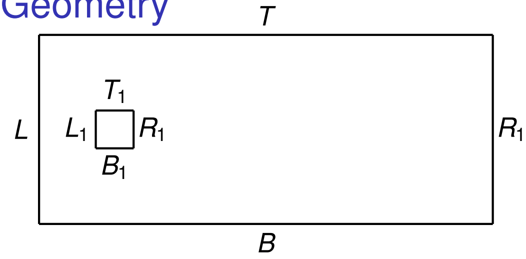
Topics

`example18.edp` Geometry




```
real L=0.0, R=2.0, B=0.0, T=1.0;
real L1=0.2, R1=0.4, B1=0.4, T1=0.6;
```

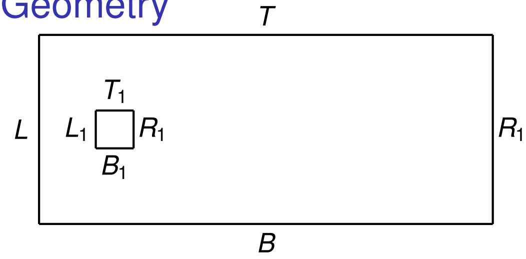
example18.edp Geometry




```
real L=0.0, R=2.0, B=0.0, T=1.0;
real L1=0.2, R1=0.4, B1=0.4, T1=0.6;


// outer square
border Bottom( t=0,1 ){x= L + (R-L)*t; y= B; label= 1;};
```

example18.edp Geometry

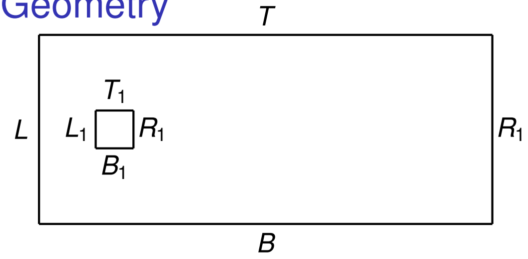



```
real L=0.0, R=2.0, B=0.0, T=1.0;
real L1=0.2, R1=0.4, B1=0.4, T1=0.6;


// outer square
border Bottom( t=0,1 ){x= L + (R-L)*t; y= B; label= 1;};

// obstacle
border ObsBottom( t=0,1 ){x= L1 + (R1-L1)*t; y= B1; label= 5;}
```

example18.edp Geometry




```
real L=0.0, R=2.0, B=0.0, T=1.0;
real L1=0.2, R1=0.4, B1=0.4, T1=0.6;
int  nx=20, ny=10, nx1=5, ny1=5;

// outer square
border Bottom( t=0,1 ){x= L + (R-L)*t; y= B; label= 1;};

// obstacle
border ObsBottom( t=0,1 ){x= L1 + (R1-L1)*t; y= B1; label= 5;}

plot( Bottom(nx) + Right(ny) + Top(-nx) + Left(-ny)
  + ObsBottom(-nx1) + ObsRight(-ny1) + ObsTop(+nx1)
  + ObsLeft(+ny1), wait=1 );
```

example18.edp Code

```
// Example 18: Vortex shedding past a square

real L= 0.0, R= 2.0, B= 0.0, T= 1.0;
real L1= 0.2, R1= 0.4, B1= 0.4, T1= 0.6;
int  nx=20, ny=10;
int  nx1=5, ny1=5;

// outer square
border Bottom( t=0,1 ){x= L + (R-L)*t; y= B; label= 1;};
border    Top( t=0,1 ){x= L + (R-L)*t; y= T; label= 3;};
border   Left( t=0,1 ){x= L; y= B + (T-B)*t; label= 4;};
border  Right( t=0,1 ){x= R; y= B + (T-B)*t; label= 2;};

// obstacle
border ObsBottom(t= 0,1){x= L1 + (R1-L1)*t; y= B1; label= 5;};
border    ObsTop(t= 0,1){x= L1 + (R1-L1)*t; y= T1; label= 7;};
border   ObsLeft(t= 0,1){x= L1; y= B1 + (T1-B1)*t; label= 8;};
border  ObsRight(t= 0,1){x= R1; y= B1 + (T1-B1)*t; label= 6;};
```

example18.edp Code

```
// Example 18: Vortex shedding past a square

real L= 0.0, R= 2.0, B= 0.0, T= 1.0;
real L1= 0.2, R1= 0.4, B1= 0.4, T1= 0.6;
int  nx=20, ny=10;
int  nx1=5, ny1=5;

// outer square
border Bottom( t=0,1 ){x= L + (R-L)*t; y= B; label= 1;};
border    Top( t=0,1 ){x= L + (R-L)*t; y= T; label= 3;};
border   Left( t=0,1 ){x= L; y= B + (T-B)*t; label= 4;};
border  Right( t=0,1 ){x= R; y= B + (T-B)*t; label= 2;};

// obstacle
border ObsBottom(t= 0,1){x= L1 + (R1-L1)*t; y= B1; label= 5;};
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border   ObsLeft(t= 0,1){x= L1; y= B1 + (T1-B1)*t; label= 8;};
border  ObsRight(t= 0,1){x= R1; y= B1 + (T1-B1)*t; label= 6;};

// plot geometry
plot( Bottom(nx) + Right(ny) + Top(-nx) + Left(-ny)
   + ObsBottom(-nx1) + ObsRight(-ny1) + ObsTop(+nx1) + ObsLeft(+ny1), wait=1 );
```

example18.edp Code

```
// Example 18: Vortex shedding past a square

real L= 0.0, R= 2.0, B= 0.0, T= 1.0;
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border    Top( t=0,1 ){x= L + (R-L)*t; y= T; label= 3;};
border   Left( t=0,1 ){x= L; y= B + (T-B)*t; label= 4;};
border  Right( t=0,1 ){x= R; y= B + (T-B)*t; label= 2;};

// obstacle
border ObsBottom(t= 0,1){x= L1 + (R1-L1)*t; y= B1; label= 5;};
border    ObsTop(t= 0,1){x= L1 + (R1-L1)*t; y= T1; label= 7;};
border   ObsLeft(t= 0,1){x= L1; y= B1 + (T1-B1)*t; label= 8;};
border  ObsRight(t= 0,1){x= R1; y= B1 + (T1-B1)*t; label= 6;};

// plot geometry
plot( Bottom(nx) + Right(ny) + Top(-nx) + Left(-ny)
   + ObsBottom(-nx1) + ObsRight(-ny1) + ObsTop(+nx1) + ObsLeft(+ny1), wait=1 );

// generate mesh
mesh Th=
```

example18.edp Code

```
// Example 18: Vortex shedding past a square

real L= 0.0, R= 2.0, B= 0.0, T= 1.0;
real L1= 0.2, R1= 0.4, B1= 0.4, T1= 0.6;
int  nx=20, ny=10;
int  nx1=5, ny1=5;

// outer square
border Bottom( t=0,1 ){x= L + (R-L)*t; y= B; label= 1;};
border    Top( t=0,1 ){x= L + (R-L)*t; y= T; label= 3;};
border   Left( t=0,1 ){x= L; y= B + (T-B)*t; label= 4;};
border  Right( t=0,1 ){x= R; y= B + (T-B)*t; label= 2;};

// obstacle
border ObsBottom(t= 0,1){x= L1 + (R1-L1)*t; y= B1; label= 5;};
border    ObsTop(t= 0,1){x= L1 + (R1-L1)*t; y= T1; label= 7;};
border   ObsLeft(t= 0,1){x= L1; y= B1 + (T1-B1)*t; label= 8;};
border  ObsRight(t= 0,1){x= R1; y= B1 + (T1-B1)*t; label= 6;};

// plot geometry
plot( Bottom(nx) + Right(ny) + Top(-nx) + Left(-ny)
   + ObsBottom(-nx1) + ObsRight(-ny1) + ObsTop(+nx1) + ObsLeft(+ny1), wait=1 );

// generate mesh
mesh Th= buildmesh( Bottom(nx) + Right(ny) + Top(-nx) + Left(-ny)
           + ObsBottom(-nx) + ObsRight(-ny) + ObsTop(+nx) + ObsLeft(+ny) );
```

example18.edp Code

```
// Example 18: Vortex shedding past a square

real L= 0.0, R= 2.0, B= 0.0, T= 1.0;
real L1= 0.2, R1= 0.4, B1= 0.4, T1= 0.6;
int  nx=20, ny=10;
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// outer square
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border   ObsLeft(t= 0,1){x= L1; y= B1 + (T1-B1)*t; label= 8;};
border  ObsRight(t= 0,1){x= R1; y= B1 + (T1-B1)*t; label= 6;};

// plot geometry
plot( Bottom(nx) + Right(ny) + Top(-nx) + Left(-ny)
   + ObsBottom(-nx1) + ObsRight(-ny1) + ObsTop(+nx1) + ObsLeft(+ny1), wait=1 );

// generate mesh
mesh Th= buildmesh( Bottom(nx) + Right(ny) + Top(-nx) + Left(-ny)
           + ObsBottom(-nx) + ObsRight(-ny) + ObsTop(+nx) + ObsLeft(+ny) );

// plot mesh
plot(Th, wait=1, ps="VortexStreetMesh.eps");
```
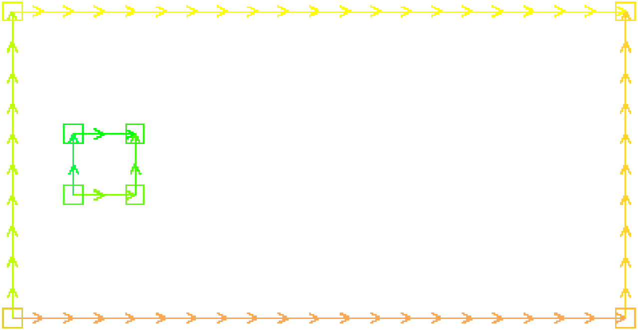
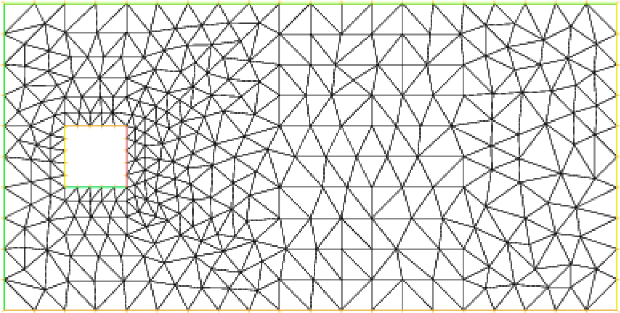

Geometry plot



Mesh plot



Topics

`example18.edp` Code, cont'd

```
fespace Xh( Th, P2);
fespace Mh( Th, P1);
```

example18.edp Code, cont'd

```
fespace Xh( Th, P2);
fespace Mh( Th, P1);
Xh u2, v2, u1, v1, up1, up2;
Mh p, q;
```

example18.edp Code, cont'd

```
fespace Xh( Th, P2);
fespace Mh( Th, P1);
Xh u2, v2, u1, v1, up1, up2;
Mh p, q;

int numTSteps=600;
bool reuseMatrix = false;
real  nu=1./1000., dt=0.05, bndryVelocity;
```

example18.edp Code, cont'd

```
fespace Xh( Th, P2);
fespace Mh( Th, P1);
Xh u2, v2, u1, v1, up1, up2;
Mh p, q;

int numTSteps=600;
bool reuseMatrix = false;
real  nu=1./1000., dt=0.05, bndryVelocity;

problem  NS ([u1, u2, p] , [v1, v2, q] , init= reuseMatrix) =
```

example18.edp Code, cont'd

```
fespace Xh( Th, P2);
fespace Mh( Th, P1);
Xh u2, v2, u1, v1, up1, up2;
Mh p, q;

int numTSteps=600;
bool reuseMatrix = false;
real  nu=1./1000., dt=0.05, bndryVelocity;

problem  NS ([u1, u2, p] , [v1, v2, q] , init= reuseMatrix) =
    int2d(Th)( 1./dt * ( u1*v1 + u2*v2 )
             + nu * ( dx(u1)*dx(v1) + dy(u1)*dy(v1)
             +        dx(u2)*dx(v2) + dy(u2)*dy(v2) )
             + p*q*(0.000001)
             + p*dx(v1) + p*dy(v2)
             + dx(u1)*q + dy(u2)*q )    MORE ...
```

example18.edp Code, cont'd

```
fespace Xh( Th, P2);
fespace Mh( Th, P1);
Xh u2, v2, u1, v1, up1, up2;
Mh p, q;

int numTSteps=600;
bool reuseMatrix = false;
real  nu=1./1000., dt=0.05, bndryVelocity;

problem  NS ([u1, u2, p] , [v1, v2, q] , init= reuseMatrix) =
    int2d(Th)( 1./dt * ( u1*v1 + u2*v2 )
             + nu * ( dx(u1)*dx(v1) + dy(u1)*dy(v1)
             +        dx(u2)*dx(v2) + dy(u2)*dy(v2) )
             + p*q*(0.000001)
             + p*dx(v1) + p*dy(v2)
             + dx(u1)*q + dy(u2)*q )
  + int2d(Th) ( -1./dt*convect( [up1, up2] , -dt , up1) * v1
                -1./dt*convect( [up1, up2] , -dt , up2) * v2 )

     MORE ...
```

example18.edp Code, cont'd

```
fespace Xh( Th, P2);
fespace Mh( Th, P1);
Xh u2, v2, u1, v1, up1, up2;
Mh p, q;

int numTSteps=600;
bool reuseMatrix = false;
real  nu=1./1000., dt=0.05, bndryVelocity;

problem  NS ([u1, u2, p] , [v1, v2, q] , init= reuseMatrix) =
    int2d(Th)( 1./dt * ( u1*v1 + u2*v2 )
             + nu * ( dx(u1)*dx(v1) + dy(u1)*dy(v1)
             +        dx(u2)*dx(v2) + dy(u2)*dy(v2) )
             + p*q*(0.000001)
             + p*dx(v1) + p*dy(v2)
             + dx(u1)*q + dy(u2)*q )
  + int2d(Th) ( -1./dt*convect( [up1, up2] , -dt , up1) * v1
                -1./dt*convect( [up1, up2] , -dt , up2) * v2 )


    // b.c.: uniform velocity top, bottom, inlet (left)
    // "do nothing" on exit (right)
  + on(1, u1=bndryVelocity, u2=0)
  + on(3, u1=bndryVelocity, u2=0)
  + on(4, u1=bndryVelocity, u2=0)
  + on(5, 6, 7, 8, u1=0, u2=0);
```

New keywords and functions

```
real
(int)
border
mesh
buildmesh
fespace
problem
int2d
dx, dy( ... )
on (for boundary conditions)
convect
```

Topics

What is `convect`?

- Section 9.5.2 of the FreeFem++ book, p. 240ff.
- Note that

$$\frac{\partial w}{\partial t} + u \cdot \nabla w = \frac{dw(X(t),t)}{dt}$$

where

$$\frac{dX(t)}{dt} = u(X(t),t)$$

What is `convect`?

- Section 9.5.2 of the FreeFem++ book, p. 240ff.
- Note that

$$\frac{\partial w}{\partial t} + u \cdot \nabla w = \frac{dw(X(t), t)}{dt}$$

  where

$$\frac{dX(t)}{dt} = u(X(t), t)$$

- Given a point x, find the solution of the “final value problem” $dX/dt = u^m(X(t))$, with $X(t^{m+1}) = x$,
- Then

$$\begin{aligned} \frac{dw(X(t), t)}{dt} &\approx \frac{w(X(t^{m+1}), t^{m+1}) - w(X(t^m), t^m)}{\Delta t} \\ &= \frac{w^{m+1} - w(X(t^m), t^m)}{\Delta t} \end{aligned}$$

What is `convect`?

- Section 9.5.2 of the FreeFem++ book, p. 240ff.
- Note that
$$\frac{\partial w}{\partial t} + u \cdot \nabla w = \frac{dw(X(t),t)}{dt}$$
where
$$\frac{dX(t)}{dt} = u(X(t),t)$$
- Given a point x, find the solution of the "final value problem" $dX/dt = u^m(X(t))$, with $X(t^{m+1}) = x$,
- Then
$$\frac{dw(X(t),t)}{dt} \approx \frac{w(X(t^{m+1}),t^{m+1}) - w(X(t^m),t^m)}{\Delta t}$$
$$= \frac{w^{m+1} - w(X(t^m),t^m)}{\Delta t}$$
- How to get $w(X(t^m),t^m)$?

The "final value problem" for **`convect`**?

- Assume $w^m(x)$ is a known function of x
- Assume $u(x,t) \approx u^m(x)$ is constant over the interval $t^m \leq t < t^{m+1}$
- $X(t^{m+1}) - X(t^m) \approx \Delta t u^m(X(t^{m+1}))$
- Solving, $x(t^m) = x - u^m(x)\Delta t$
- $w(X(t^m), t^m) = w(x - u^m \Delta t)$
- **`convect(u,-dt,w)`** $(x) = w(x - u\,dt)$

That sounds great! Why doesn't everyone do it?

- Those approximations are $O(\Delta t)$
- What if the flow is fast enough so $x - u\,dt$ is out of the element?
- What if the flow is fast enough so $x - u\,dt$ is out of Ω?
- Hard to prove things about.
- Stability?

Topics

More syntax

- Declare a real vector: **`real[int] v(100);`**
- Looping

```
for (int index=0 ; index < maximum ; i++) {
  ... code ...
}
```

- Can use an already-declared index variable.
- **`index`** is no longer “in scope” after the loop ends

Scoping, blocking and variable declarations

- Variables can be declared anywhere in the program
- “Blocks” of code are inside curly brackets: `{ ... }`
- Declarations inside a block:
  - Are not known outside the block.
  - “Cover up” previous declarations

Asking questions: `if`

```
boolean expression ? true expression : false expression
```

```
if ( boolean expression ){
   ...  code  ...
}
```

```
if ( boolean expression ){
   ...  code  ...
} else if {
   ...  code  ...
} else {
   ...  code  ...
}
```

A single statement does not need to be inside a block.

Boolean expressions

Comparison operators	
operator	description
`==`	equal to
`!=`	not equal to
`<`	less than
`<=`	less than or equal to
`>=`	greater than or equal to

Logical operators	
operator	description
`!`	not
`&&`	and
`||`	or

Do not use `&` *or* `|`! *They are bitwise operators.*

Remark: Logical operators are “short circuit”

example18.edp Code, cont'd

```
real[int] times(numTSteps), var(numTSteps);

for (int i=0 ; i < numTSteps ; i++){
  times[i] = i*dt;







  up1 = u1;
  up2 = u2;

  // Solve the problem
  NS;

  // reuse the matrix in the rest of the iterations
  reuseMatrix = true

  // plot current solution
  plot(coef=0.2, cmm= " [u1,u2] and p  ", p, [u1,u2]
    ArrowShape= 1, ArrowSize= -0.8);
}
```

example18.edp Code, cont'd

```
real[int] times(numTSteps), var(numTSteps);

for (int i=0 ; i < numTSteps ; i++){
  times[i] = i*dt;







  up1 = u1;
  up2 = u2;

  // Solve the problem
  NS;

  // reuse the matrix in the rest of the iterations
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  plot(coef=0.2, cmm= " [u1,u2] and p  ", p, [u1,u2]
    ArrowShape= 1, ArrowSize= -0.8);
}
```

example18.edp Code, cont'd

```
real[int] times(numTSteps), var(numTSteps);

for (int i=0 ; i < numTSteps ; i++){
  times[i] = i*dt;

  // ramp up velocity from 0.0
  bndryVelocity = i*dt;
  if (bndryVelocity >= 1.0){
    bndryVelocity = 1.0;
  }

  up1 = u1;
  up2 = u2;

  // Solve the problem
  NS;

  // reuse the matrix in the rest of the iterations
  reuseMatrix = true

  // plot current solution
  plot(coef=0.2, cmm= " [u1,u2] and p  ", p, [u1,u2]
    ArrowShape= 1, ArrowSize= -0.8);
}
```

Exercise 22 (10 points)

Example 18 presents a vortex-shedding problem using the "**`convect`**" function in FreeFem++. Convert the code to use the conventional form of the convection terms. You should observe vortex shedding behavior in your formulation, although details may differ. Explain any differences you think are important between the two solutions.

Comparison with FEniCS code: boundary conditions

- FreeFem++ code goes into the weak form

```
+ on(1, u1=bndryVelocity, u2=0)
+ on(3, u1=bndryVelocity, u2=0)
+ on(4, u1=bndryVelocity, u2=0)
+ on(5, 6, 7, 8, u1=0, u2=0)
```

- FEniCS code requires a mesh function

```
boundaries = MeshFunction("size_t", mesh, mesh.topology().dim()-1)
```

- FEniCS code requires new classes

```
class InflowBoundary(SubDomain):
    def inside(self, x, on_boundary):
        return on_boundary and x[0] < xmin + bmarg
```

- FEniCS requires instantiation

```
inflowBoundary = InflowBoundary()
g1 = Expression( ("4.*Um*(x[1]*(ymax-x[1]))/(ymax*ymax)" , "0.0"), \
                                  Um=1.5, ymax=ymax)
bc1 = DirichletBC(W.sub(0), g1, inflowBoundary)
```

- FEniCS then has the b.c.s passed to the solver with the matrices.

```
solve(NSE == LNSE, w, bcs)
```

Comparison with FEniCS code: functions and spaces

- FEniCS:

```
V = VectorFunctionSpace(mesh, "CG", 2)
Q = FunctionSpace(mesh, "CG", 1)
W = V * Q
   ...
(u, p) = TrialFunctions(W)
(v, q) = TestFunctions(W)
w = Function(W)
u0 = Function(V)
```

- FreeFem++

```
fespace Xh( Th, P2);
fespace Mh( Th, P1);
Xh u2, v2, u1, v1, up1, up2;
Mh p, q;
```

Comparison with FEniCS code: Weak form

- FEniCS

```
LNSE = inner(u0,v)*dx
NSE = (inner(u,v) + dt*(inner(grad(u)*u0,v) \
              + nu*inner(grad(u),grad(v)) - div(v)*p) + q*div(u) )*dx
```

- FreeFem++

```
problem  NS ([u1, u2, p] , [v1, v2, q] =
    int2d(Th)( 1./dt*( u1*v1 + u2*v2 )
             + nu * ( dx(u1)*dx(v1) + dy(u1)*dy(v1)
             +        dx(u2)*dx(v2) + dy(u2)*dy(v2) )
             + p*q*(0.000001)
             + p*dx(v1) + p*dy(v2)
             + dx(u1)*q + dy(u2)*q )
  + int2d(Th) ( -1./dt*convect( [up1, up2] , -dt , up1) * v1
                -1./dt*convect( [up1, up2] , -dt , up2) * v2 )
  plus b.c.s
```

I'd like to print some stuff.

- Printing uses **cout** and **<<**
- Add lines inside the loop

```
cout << "t=" << times[i] << " umax="
     << max(u1[].max, u2[].max)
     << " vertical velocity=" << u2(0.5, 0.5) << endl;
```

Can I put some stuff into a file?

- Place file definition and use inside a block
- File is closed when it goes out of scope

```
{
ofstream vels("vels.txt");
for (i=0 ; i < numTSteps ; i++) {

   // ramp up velocity from 0.0
   bndryVelocity = i*dt;
   if (bndryVelocity >= 1.0){
     bndryVelocity = 1.0;
   }

   up1 = u1;
   up2 = u2;

   // solve the problem
   NS;

   // reuse the matrix in the rest of the iterations
   reuseMatrix = true;

   // write a 2-column file (t,velocity)
   vels << i*dt << " " << u2(0.5, 0.5) << endl;
}
} // file is closed
```

Topics

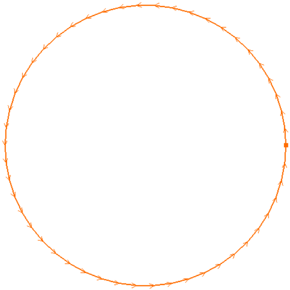
`example19.edp` Poisson's equation on a circle

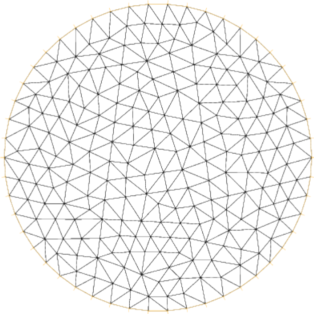
$$
\begin{aligned}
-\Delta u(x,y) &= f(x,y) \text{ inside } \Omega \\
u(x,y) &= 0 \text{ on } \partial\Omega
\end{aligned}
$$

where Ω is the unit circle in $\mathbb{R}^2$.

example19.edp code

```
// defining the boundary
border C(t=0,2*pi){x=cos(t); y=sin(t);}
// the triangulated domain Th is on the left side of its boundary
// because boundary parameterization is CCW

// mesh based on 50 t-increments
mesh Th = buildmesh (C(50));
```





Remark: Mesh boundary is polygonal, not curved. P^2 elements will have extra boundary nodes off the circle.

example19.edp code

```
                // the finite element space defined over Th is called here Vh
fespace Vh(Th,P1);
```

example19.edp code

```
                // the finite element space defined over Th is called here Vh
fespace Vh(Th,P1);

Vh u,v;                // defines u and v as piecewise-P1 continuous functions
```

example19.edp code

```
                 // the finite element space defined over Th is called here Vh
fespace Vh(Th,P1);

Vh u,v;                // defines u and v as piecewise-P1 continuous functions

func f= x*y;                       // definition of a function named f for RHS
```

example19.edp code

```
                // the finite element space defined over Th is called here Vh
fespace Vh(Th,P1);

Vh u,v;                // defines u and v as piecewise-P1 continuous functions

func f= x*y;                           // definition of a function named f for RHS

real cpu = clock();                                         // get the clock in second
```

example19.edp code

```
                // the finite element space defined over Th is called here Vh
fespace Vh(Th,P1);

Vh u,v;               // defines u and v as piecewise-P1 continuous functions

func f= x*y;                             // definition of a function named f for RHS

real cpu = clock();                                        // get the clock in second

solve Poisson(u, v, solver=UMFPACK) =                                // defines the PDE
  int2d(Th)( dx(u)*dx(v) + dy(u)*dy(v) )                             // bilinear part
  - int2d(Th)( f*v )                                                 // right hand side
   + on(C, u=0) ;                                        // Dirichlet boundary condition
```

example19.edp code

```
                    // the finite element space defined over Th is called here Vh
fespace Vh(Th,P1);

Vh u,v;               // defines u and v as piecewise-P1 continuous functions

func f= x*y;                          // definition of a function named f for RHS

real cpu = clock();                                     // get the clock in second

solve Poisson(u, v, solver=UMFPACK) =                           // defines the PDE
  int2d(Th)( dx(u)*dx(v) + dy(u)*dy(v) )                          // bilinear part
  - int2d(Th)( f*v )                                            // right hand side
   + on(C, u=0) ;                                   // Dirichlet boundary condition

plot(u, ps="solution19.eps");                                     // plot solution
```

example19.edp code

```
                // the finite element space defined over Th is called here Vh
fespace Vh(Th,P1);

Vh u,v;                // defines u and v as piecewise-P1 continuous functions

func f= x*y;                       // definition of a function named f for RHS

real cpu = clock();                                 // get the clock in second

solve Poisson(u, v, solver=UMFPACK) =                       // defines the PDE
  int2d(Th)( dx(u)*dx(v) + dy(u)*dy(v) )                     // bilinear part
  - int2d(Th)( f*v )                                        // right hand side
   + on(C, u=0) ;                               // Dirichlet boundary condition

plot(u, ps="solution19.eps");                                // plot solution

cout << " CPU time = " << clock()-cpu << endl;           // print time required
```

Topics

Some more syntax

- Book Chapter 4
- Variables: **`real`**, **`int`**, **`bool`**, **`complex`**
- Variable names: letters and numbers, no _
- File variables: **`ofstream`**, **`ifstream`**
- Global variables: **`cout`**, **`cin`**, **`true`**, **`false`**, **`pi`**, **`i`**
- Arrays: **`real[int]`**

Global variables (used mainly in weak forms)

- For current point
  - **x**, **y**, **z**
  - **label** (boundary point label, 0 if not boundary)
  - **region**
  - **P**, **P.x**, **P.y**, **P.z**
  - **N**, **N.x**, **N.y**, **N.z**
- **lenEdge** length of current edge
- **hTriangle** size of current triangle
- **area** area of current triangle
- **volume** volume of current triangle
- **nuTriangle** number (int) of current triangle
- **nuEdge** number (int) of current edge
- **nuTonEdge** number (int) of *adjacent triangle*

Operations and functions

- Usual arithmetic operators
- Raise to power ^
- Usual elementary functions (**`sin`**, **`acosh`**, *etc.*)
- **`randres53`** generates uniform reals in $[0, 1)$ with 53-bit resolution

Functions

- **func** declares a function
- Simple example

```
func u = x2̂ + sin(pi*y)
```

- With declarations

```
func real u( int k){
  return 3.0*k*k;
}
```

- With arrays

```
func real[int] sqarr( int[int] L, int n){
  real[int] ans(n);
  for (int k=0; k<n ; k++){
    ans[k]=L[k]^2;
  }
  return ans;
}
```

- Examples → **tutorial/func.edp**

Exercise 23 (5 points)

The following example is presented in Chapter 4.

```
mesh Th=square(20,20,[-pi+2*pi*x,-pi+2*pi*y]);      // [-pi,pi] X [-pi,pi]
fespace Vh(Th,P2);
func z=x+y*1i;                                      // x+iy
func f=imag(sqrt(z));
func g=abs( sin(z/10)*exp(z^2/10) );                // complex arguments
Vh fh = f;
plot(fh);                                           // contours of f
Vh gh = g;
plot(gh);                                           // contours of g
```

- Copy this to a file, run FreeFem++, and send me the plots.
- Change the mesh so that the base of the square is oriented at an angle of $\pi/4$ instead of being horizontal. Send me the changed code and the resulting plots.

Array operations

```
int i;
real [int] tab(10), tab1(10); // 2 array of 10 real
complex [int] ctab(10), ctab1(10); // 2 array of 10 complex

tab     = 1;  //  set all the array to 1
tab[1]  = 2;
ctab    = 1+2i;  //  set all the array to 1+2i
ctab[1] = 2;
cout << "tab:  " << tab[1] << " " << tab[9] << " size = " << tab.n << endl;
cout << "ctab: " << ctab[1] << " " << ctab[9] << " size = " << ctab.n << endl;
```

Yields as output

```
tab:  2 1 size = 10
ctab: (2,0) (1,2) size = 10
```

Array operations, cont'd

```
tab1 = tab;
tab  = tab + tab1;
tab  = 2*tab + tab1*5;
tab1 = 2*tab - tab1*5;
tab += tab;
cout << "whole array tab = " << tab << endl;
cout << "tab[1] = " << tab[1] << " tab[9] = " << tab[9] <<  endl;
```

Yields as output

```
whole array tab = 10
         18       36       18       18       18
         18       18       18       18       18

tab[1] = 36 tab[9] = 18
```

Array operations, cont'd

```
ctab1 = ctab;
ctab  = ctab + ctab1;
ctab  = 2*ctab + ctab1*5;
ctab1 = 2*ctab - ctab1*5;
ctab += ctab;
cout << "whole array ctab = " << ctab << endl;
cout << "ctab[1] = " << ctab[1] << " ctab[9] = " << ctab[9] <<  endl << endl;
```

Yields as output

```
whole array ctab = 10
        (18,36) (36,0)  (18,36) (18,36) (18,36)
        (18,36) (18,36) (18,36) (18,36) (18,36)

ctab[1] = (36,0) ctab[9] = (18,36)
```

Array operations, cont'd

```
real [string] map; //  a dynamically-sized array
map["1"] = 2.0;
map[2] = 3.0; //  2 is automaticaly cast to the string "2"

cout << " map[1̈]̈ = " << map["1"] << " == 2.0 ; "<< endl;
cout << " map[2] = " << map[2] << " == 3.0 "<< endl;
assert( abs(map["1"] - 2.0) < 1.e-6);
assert( abs(map[1]   - 2.0) < 1.e-6);
```

Yields as output

```
 map["1"] = 2 == 2.0 ;
 map[2] = 3 == 3.0
```

Array operations, cont'd

```
real [string] map; //  a dynamically-sized array
map["1"] = 2.0;
map[2] = 3.0; //  2 is automaticaly cast to the string "2"

cout << " map[ïï] = " << map["1"] << " == 2.0 ; "<< endl;
cout << " map[2] = " << map[2] << " == 3.0 "<< endl;
assert( abs(map["1"] - 2.0) < 1.e-6);
assert( abs(map[1]   - 2.0) < 1.e-6);
assert( abs(map[2]   - 3.0) < 1.e-6);
```

Yields as output

```
 map["1"] = 2 == 2.0 ;
 map[2] = 3 == 3.0
```

Array operations, cont'd

```
real [int] tab2  = [1,2,3,3.14];
int  [int] itab2 = [1,2,3,5];

cout << "Length of array tab2 = " << tab2.n << endl;
cout << "Whole array tab2 = " << tab2 << endl;
cout << "Whole array itab2 = " << itab2 << endl;
```

```
tab2 /= 2;
cout << "Whole array tab2/2 = " << tab2 << endl;
tab2 *= 2;
cout << "Whole array tab2*2 = " << tab2 << endl;
```

Array operations, cont'd

```
real [int] tab2  = [1,2,3,3.14];
int  [int] itab2 = [1,2,3,5];

cout << "Length of array tab2 = " << tab2.n << endl;
cout << "Whole array tab2 = " << tab2 << endl;
cout << "Whole array itab2 = " << itab2 << endl;

tab2.resize(10);
for (int i=4; i<tab2.n; i++){
  tab2[i]=i;
}
cout << "Whole resized array tab2 = " << tab2 << endl;
tab2 /= 2;
cout << "Whole array tab2/2 = " << tab2 << endl;
tab2 *= 2;
cout << "Whole array tab2*2 = " << tab2 << endl;
```

Output from previous operations, cont'd

```
Whole array tab2 = 4
          1        2        3     3.14
Whole array itab2 = 4
          1        2        3        5
Whole resized array tab2 = 10
          1        2        3     3.14        4
          5        6        7        8        9

Whole array tab2/2 = 10
        0.5        1      1.5     1.57        2
        2.5        3      3.5        4      4.5

Whole array tab2*2 = 10
          1        2        3     3.14        4
          5        6        7        8        9
```

Array operations: 2D arrays

```
real[int,int] mat(5,5),mmat(5,5);
mat=0;
for(int i=0; i< mat.n; i++){
  for(int j=0; j< mat.m; j++){
    mat(i,j) = i + 100*(j + 1);
  }
}
mmat=mat;
cout << "mmat 2D array = " << mmat << endl;

mat.resize(10,10);
// add new rows
for(int i=5; i<mat.n ;i++){
  for(int j=0; j<mat.m ;j++){
    mat(i,j) = i + 100*(j + 1);
  }
}
// add new columns
for(int i=0;i<mat.n;i++){
  for(int j=5;j<mat.m;j++){
    mat(i,j) = i + 100*(j + 1);
  }
}
cout << "Expanded mat array = " << mat << endl;
```

Output from previous operations, cont'd

```
mmat 2D array = 5 5
         100 200 300 400 500
         101 201 301 401 501
         102 202 302 402 502
         103 203 303 403 503
         104 204 304 404 504

Expanded mat array = 10 10
         100 200 300 400 500 600 700 800 900 1000
         101 201 301 401 501 601 701 801 901 1001
         102 202 302 402 502 602 702 802 902 1002
         103 203 303 403 503 603 703 803 903 1003
         104 204 304 404 504 604 704 804 904 1004
         105 205 305 405 505 605 705 805 905 1005
         106 206 306 406 506 606 706 806 906 1006
         107 207 307 407 507 607 707 807 907 1007
         108 208 308 408 508 608 708 808 908 1008
         109 209 309 409 509 609 709 809 909 1009
```

Array operations: 1D array of mesh

```
mesh[int] aTh(10);
aTh[1]= square(2,2);
plot(aTh[1]);
aTh[2]= square(3,4);
plot(aTh[2]);
```

Topics

Topics

Topics

Example 20: Section 3.1

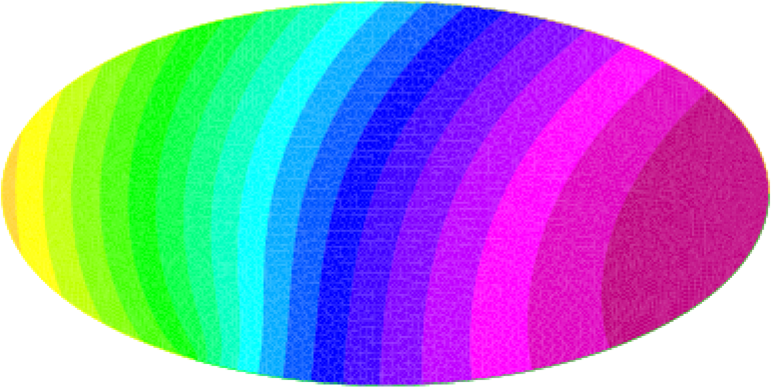
$$-\Delta\phi = f \text{ in } \Omega$$

- Elastic membrane Ω
- Rigid support Γ, may have vertical displacement
- Γ is ellipse
- Load f
- Solving for vertical displacement, ϕ
- Membrane glued to Γ: Dirichlet b.c.
- Membrane free at Γ: Neumann b.c.
- New:
  - Both Dirichlet and Neumann b.c.
  - Accessing values from mesh and solution
  - Write a plot file for **`gnuplot`**

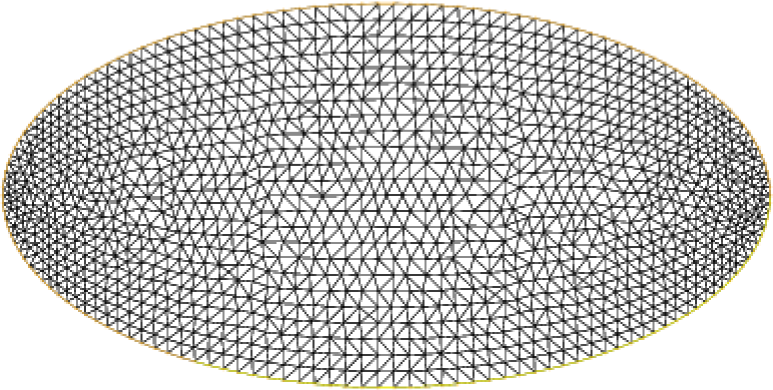
example20.edp code

```
// example20.edp
// original file: membrane.edp

real theta = 4.*pi/3.;
real a = 2.,b = 1.; // semimajor and semiminor axes

func bndryelev = x;  // elevation of Gamma1 boundary

border Gamma1(t=0, theta)     { x = a * cos(t); y = b*sin(t); }
border Gamma2(t=theta, 2*pi) { x = a * cos(t); y = b*sin(t); }
mesh Th = buildmesh( Gamma1(100) + Gamma2(50) );   // construct mesh

fespace Vh(Th,P2); // P2 conforming triangular FEM
Vh phi,w, f=1;     // phi is shape function, w is test function, f is load

// problem definition
solve Laplace(phi, w) = int2d(Th)( dx(phi)*dx(w) + dy(phi)*dy(w) )
                     - int2d(Th)( f*w ) + on(Gamma1, phi= bndryelev);

plot(phi, wait=true, ps="solution20.eps"); //Plot solution

plot(Th, wait=true, ps="mesh20.eps"); //Plot mesh
```

Plot solution



Plot mesh



`gnuplot` file

- Gnuplot file has groups of 4 lines, each with a vertex location and value x_j, y_j, ϕ_j for $j = 0, 1, 2, 0$ followed by a blank line.
- Commands for **`gnuplot`** are

```
set palette rgbformulae 30,31,32
splot "graph.txt" with lines palette
```

example20.edp code cont'd

```
// to build a gnuplot data file
{ ofstream ff("graph20.txt");

   for (int i=0;i<Th.nt;i++){
     for (int j=0; j <3; j++){
       ff << Th[i][j].x <<"    "<< Th[i][j].y <<"  "<< phi[][Vh(i,j)] << endl;
     }
     ff << Th[i][0].x << "    " << Th[i][0].y<< "  " << phi[][Vh(i,0)]
        << endl << endl << endl;
   }
}
```

Th.nt is number of triangles in $\mathcal{T}_h$.

`example20.edp` code cont'd

```
// to build a gnuplot data file
{ ofstream ff("graph20.txt");

   for (int i=0;i<Th.nt;i++){
     for (int j=0; j <3; j++){
       ff << Th[i][j].x <<"    "<< Th[i][j].y <<"  "<< phi[][Vh(i,j)] << endl;
     }
     ff << Th[i][0].x << "    " << Th[i][0].y<< "  " << phi[][Vh(i,0)]
        << endl << endl << endl;
   }
}
```

`Th[i][j].x` is x-coordinate of vertex `j` of triangle `i` in the mesh

`example20.edp` code cont'd

```
// to build a gnuplot data file
{ ofstream ff("graph20.txt");

   for (int i=0;i<Th.nt;i++){
     for (int j=0; j <3; j++){
       ff << Th[i][j].x <<"    "<< Th[i][j].y <<"  "<< phi[][Vh(i,j)] << endl;
     }
     ff << Th[i][0].x << "    " << Th[i][0].y<< "  " << phi[][Vh(i,0)]
         << endl << endl << endl;
   }
}
```

`phi[][Vh(i,j)]` is the value of dof of ϕ located at vertex `j` of triangle `i` in `fespace` V_h. The extra dofs have numbers 3 and higher.

Demonstration (for Example20)

```
$ gnuplot

gnuplot> set palette rgbformulae 30,31,32
gnuplot> splot "graph.txt" with lines palette
```

Topics

Example 21: errors

- from **`membranerror.edp`**
- Like Example 20
- Change to have exact solution
- Look at errors and convergence
- New:
  - Turning off extraneous output
  - Plot **`Th`** and **`phi`** together
  - Loading extra elements

example21.edp code

```
verbosity=0;

load "Element_P3"

real theta = 4.*pi/3.;
real a=1., b=1.; // ellipse is a circle here
border Gamma1(t=0.0,theta)    { x = a * cos(t); y = b*sin(t); }
border Gamma2(t=theta,2.0*pi) { x = a * cos(t); y = b*sin(t); }

func f=-4.0*(cos(x^2+y^2-1.0) -(x^2+y^2)*sin(x^2+y^2-1.0));
func phiexact=sin(x^2+y^2-1.0);

int meshes=3, mshdensity=1;
real[int] L2error(meshes);
for ( int n=0; n<meshes; n++){
  mesh Th = buildmesh( Gamma1(40*mshdensity) + Gamma2(20*mshdensity) );
  mshdensity *= 2;
  fespace Vh(Th,P2);
  Vh phi,w;

  solve laplace(phi, w, solver=UMFPACK) =
    int2d(Th)( dx(phi) * dx(w) + dy(phi) * dy(w))
    - int2d(Th)( f*w ) + on(Gamma2, phi=0)+ on(Gamma1, phi=0);

  phi = (phi-phiexact);
  plot(Th, phi, wait=true, fill=true); //Plot Th and phi

  // compute error
  L2error[n]= sqrt( int2d(Th)( (phi)^2 ) );
}
```

example21.edp code

```
verbosity=0;

load "Element_P3"

real theta = 4.*pi/3.;
real a=1., b=1.; // ellipse is a circle here
border Gamma1(t=0.0,theta)    { x = a * cos(t); y = b*sin(t); }
border Gamma2(t=theta,2.0*pi) { x = a * cos(t); y = b*sin(t); }

func f=-4.0*(cos(x^2+y^2-1.0) -(x^2+y^2)*sin(x^2+y^2-1.0));
func phiexact=sin(x^2+y^2-1.0);

int meshes=3, mshdensity=1;
real[int] L2error(meshes);
for ( int n=0; n<meshes; n++){
  mesh Th = buildmesh( Gamma1(40*mshdensity) + Gamma2(20*mshdensity) );
  mshdensity *= 2;
  fespace Vh(Th,P2);
  Vh phi,w;

  solve laplace(phi, w, solver=UMFPACK) =
    int2d(Th)( dx(phi) * dx(w) + dy(phi) * dy(w))
    - int2d(Th)( f*w ) + on(Gamma2, phi=0)+ on(Gamma1, phi=0);

  phi = (phi-phiexact);
  plot(Th, phi, wait=true, fill=true); //Plot Th and phi

  // compute error
  L2error[n]= sqrt( int2d(Th)( (phi)^2 ) );
}
```

example21.edp code

```
verbosity=0;

load "Element_P3"

real theta = 4.*pi/3.;
real a=1., b=1.; // ellipse is a circle here
border Gamma1(t=0.0,theta)    { x = a * cos(t); y = b*sin(t); }
border Gamma2(t=theta,2.0*pi) { x = a * cos(t); y = b*sin(t); }

func f=-4.0*(cos(x^2+y^2-1.0) -(x^2+y^2)*sin(x^2+y^2-1.0));
func phiexact=sin(x^2+y^2-1.0);

int meshes=3, mshdensity=1;
real[int] L2error(meshes);
for ( int n=0; n<meshes; n++){
  mesh Th = buildmesh( Gamma1(40*mshdensity) + Gamma2(20*mshdensity) );
  mshdensity *= 2;
  fespace Vh(Th,P2);
  Vh phi,w;

  solve laplace(phi, w, solver=UMFPACK) =
    int2d(Th)( dx(phi) * dx(w) + dy(phi) * dy(w))
    - int2d(Th)( f*w ) + on(Gamma2, phi=0)+ on(Gamma1, phi=0);

  phi = (phi-phiexact);
  plot(Th, phi, wait=true, fill=true); //Plot Th and phi

  // compute error
  L2error[n]= sqrt( int2d(Th)( (phi)^2 ) );
}
```

example21.edp code

```
verbosity=0;

load "Element_P3"

real theta = 4.*pi/3.;
real a=1., b=1.; // ellipse is a circle here
border Gamma1(t=0.0,theta)    { x = a * cos(t); y = b*sin(t); }
border Gamma2(t=theta,2.0*pi) { x = a * cos(t); y = b*sin(t); }

func f=-4.0*(cos(x^2+y^2-1.0) -(x^2+y^2)*sin(x^2+y^2-1.0));
func phiexact=sin(x^2+y^2-1.0);

int meshes=3, mshdensity=1;
real[int] L2error(meshes);
for ( int n=0; n<meshes; n++){
  mesh Th = buildmesh( Gamma1(40*mshdensity) + Gamma2(20*mshdensity) );
  mshdensity *= 2;
  fespace Vh(Th,P2);
  Vh phi,w;

  solve laplace(phi, w, solver=UMFPACK) =
    int2d(Th)( dx(phi) * dx(w) + dy(phi) * dy(w))
    - int2d(Th)( f*w ) + on(Gamma2, phi=0)+ on(Gamma1, phi=0);

  phi = (phi-phiexact);
  plot(Th, phi, wait=true, fill=true); //Plot Th and phi

  // compute error
  L2error[n]= sqrt( int2d(Th)( (phi)^2 ) );
}
```

example21.edp code

```
verbosity=0;

load "Element_P3"

real theta = 4.*pi/3.;
real a=1., b=1.; // ellipse is a circle here
border Gamma1(t=0.0,theta)    { x = a * cos(t); y = b*sin(t); }
border Gamma2(t=theta,2.0*pi) { x = a * cos(t); y = b*sin(t); }

func f=-4.0*(cos(x^2+y^2-1.0) -(x^2+y^2)*sin(x^2+y^2-1.0));
func phiexact=sin(x^2+y^2-1.0);

int meshes=3, mshdensity=1;
real[int] L2error(meshes);
for ( int n=0; n<meshes; n++){
  mesh Th = buildmesh( Gamma1(40*mshdensity) + Gamma2(20*mshdensity) );
  mshdensity *= 2;
  fespace Vh(Th,P2);
  Vh phi,w;

  solve laplace(phi, w, solver=UMFPACK) =
    int2d(Th)( dx(phi) * dx(w) + dy(phi) * dy(w))
    - int2d(Th)( f*w ) + on(Gamma2, phi=0)+ on(Gamma1, phi=0);

  phi = (phi-phiexact);
  plot(Th, phi, wait=true, fill=true); //Plot Th and phi

  // compute error
  L2error[n]= sqrt( int2d(Th)( (phi)^2 ) );
}
```

example21.edp code

```
verbosity=0;

load "Element_P3"

real theta = 4.*pi/3.;
real a=1., b=1.; // ellipse is a circle here
border Gamma1(t=0.0,theta)    { x = a * cos(t); y = b*sin(t); }
border Gamma2(t=theta,2.0*pi) { x = a * cos(t); y = b*sin(t); }

func f=-4.0*(cos(x^2+y^2-1.0) -(x^2+y^2)*sin(x^2+y^2-1.0));
func phiexact=sin(x^2+y^2-1.0);

int meshes=3, mshdensity=1;
real[int] L2error(meshes);
for ( int n=0; n<meshes; n++){
  mesh Th = buildmesh( Gamma1(40*mshdensity) + Gamma2(20*mshdensity) );
  mshdensity *= 2;
  fespace Vh(Th,P2);
  Vh phi,w;

  solve laplace(phi, w, solver=UMFPACK) =
    int2d(Th)( dx(phi) * dx(w) + dy(phi) * dy(w))
    - int2d(Th)( f*w ) + on(Gamma2, phi=0)+ on(Gamma1, phi=0);

  phi = (phi-phiexact);
  plot(Th, phi, wait=true, fill=true); //Plot Th and phi

  // compute error
  L2error[n]= sqrt( int2d(Th)( (phi)^2 ) );
}
```

example21.edp code

```
verbosity=0;

load "Element_P3"

real theta = 4.*pi/3.;
real a=1., b=1.; // ellipse is a circle here
border Gamma1(t=0.0,theta)    { x = a * cos(t); y = b*sin(t); }
border Gamma2(t=theta,2.0*pi) { x = a * cos(t); y = b*sin(t); }

func f=-4.0*(cos(x^2+y^2-1.0) -(x^2+y^2)*sin(x^2+y^2-1.0));
func phiexact=sin(x^2+y^2-1.0);

int meshes=3, mshdensity=1;
real[int] L2error(meshes);
for ( int n=0; n<meshes; n++){
  mesh Th = buildmesh( Gamma1(40*mshdensity) + Gamma2(20*mshdensity) );
  mshdensity *= 2;
  fespace Vh(Th,P2);
  Vh phi,w;

  solve laplace(phi, w, solver=UMFPACK) =
    int2d(Th)( dx(phi) * dx(w) + dy(phi) * dy(w))
    - int2d(Th)( f*w ) + on(Gamma2, phi=0)+ on(Gamma1, phi=0);

  phi = (phi-phiexact);
  plot(Th, phi, wait=true, fill=true); //Plot Th and phi

  // compute error
  L2error[n]= sqrt( int2d(Th)( (phi)^2 ) );
}
```

example21.edp code cont'd

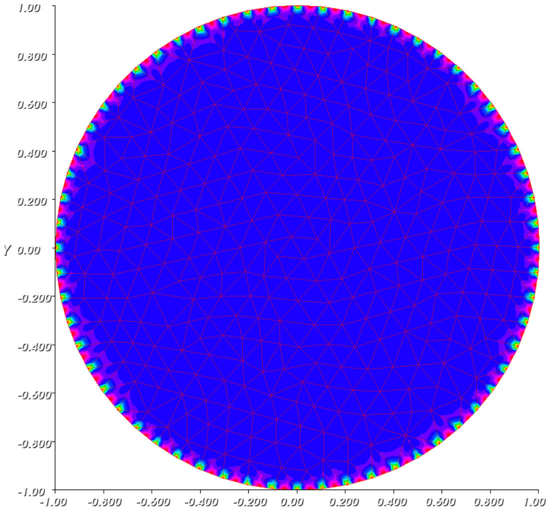
```
// print errors
for (int n=0; n<meshes; n++){
  cout << " L2error " << n << " = "<<  L2error[n] <<endl;
}

// print convergence rates
for (int n=1; n<meshes; n++){
  cout <<" convergence rate = "<< log( L2error[n-1] / L2error[n] )/log(2.)
       <<endl;
}
```

Output

```
L2error 1 = 0.000816958
L2error 2 = 0.000203404
L2error 3 = 5.07361e-05
convergence rate = 2.01175
convergence rate = 2.00592
convergence rate = 2.00326
```

Error rates too slow!

Error on coarsest mesh



Maximum errors are near boundary because of geometry errors.

Change boundary condition

Change from

```
on(Gamma2, phi=0)
```

to

```
on(Gamma2, phi=phiexact)
```

Output:

```
L2error 1 = 8.05871e-08
L2error 2 = 5.4554e-09
L2error 3 = 3.30556e-10
convergence rate = 4.31729
convergence rate = 3.88479
convergence rate = 4.04472
```

WARNING: special elements need FreeFem++-cs

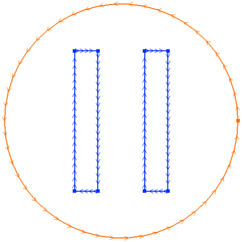
LD_LIBRARY_PATH needs proper definition

Topics

Topics

Example 22: Heat exchanger

- Circular enclosure C_0 containing two rectangular thermal conductors C_1 and C_2
- C_1 held at constant temperature
- C_2 has higher thermal conductivity.
- $\nabla \cdot (\kappa \nabla u) = 0$ on Ω with $u|_\Gamma = g$
- New stuff
  - More complex geometry
  - Saving and retrieving the mesh



example22.edp code

```
// Either build mesh or retrieve mesh
mesh Th;
int C0, C1, C2;
```

example22.edp code

```
// Either build mesh or retrieve mesh
mesh Th;
int C0, C1, C2;
if (true) {
  C0= 99; C1= 98; C2= 97; // could be anything
  border C00( t=0,2*pi ){ x=5*cos(t); y=5*sin(t); label=C0;} \ outer

  border C11(t=0,1){ x=1+t;  y=3;      label=C1;} \ heated blade
  border C12(t=0,1){ x=2;    y=3-6*t;  label=C1;}
  border C13(t=0,1){ x=2-t;  y=-3;     label=C1;}
  border C14(t=0,1){ x=1;    y=-3+6*t; label=C1;}

  border C21(t=0,1){ x=-2+t; y=3;      label=C2;} \ cooling blade
  border C22(t=0,1){ x=-1;   y=3-6*t;  label=C2;}
  border C23(t=0,1){ x=-1-t; y=-3;     label=C2;}
  border C24(t=0,1){ x=-2;   y=-3+6*t; label=C2;}
```

example22.edp code

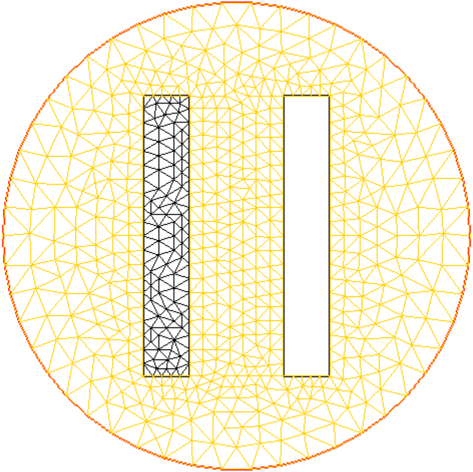
```
// Either build mesh or retrieve mesh
mesh Th;
int C0, C1, C2;
if (true) {
  C0= 99; C1= 98; C2= 97; // could be anything
  border C00( t=0,2*pi ){ x=5*cos(t); y=5*sin(t); label=C0;} \ outer

  border C11(t=0,1){ x=1+t;  y=3;      label=C1;} \ heated blade
  border C12(t=0,1){ x=2;    y=3-6*t;  label=C1;}
  border C13(t=0,1){ x=2-t;  y=-3;     label=C1;}
  border C14(t=0,1){ x=1;    y=-3+6*t; label=C1;}

  border C21(t=0,1){ x=-2+t; y=3;      label=C2;} \ cooling blade
  border C22(t=0,1){ x=-1;   y=3-6*t;  label=C2;}
  border C23(t=0,1){ x=-1-t; y=-3;     label=C2;}
  border C24(t=0,1){ x=-2;   y=-3+6*t; label=C2;}

            plot(    C00(50)
                   + C11( 5) + C12( 20) + C13( 5) + C14( 20)
                   + C21(-5) + C22(-20) + C23(-5) + C24(-20), wait=true);

  Th = buildmesh(    C00(50)
                   + C11( 5) + C12( 20) + C13( 5) + C14( 20)
                   + C21(-5) + C22(-20) + C23(-5) + C24(-20));

  plot(Th,wait=true);
```

example22.edp code

```
// Either build mesh or retrieve mesh
mesh Th;
int C0, C1, C2;
if (true) {
  C0= 99; C1= 98; C2= 97; // could be anything
  border C00( t=0,2*pi ){ x=5*cos(t); y=5*sin(t); label=C0;} \ outer

  border C11(t=0,1){ x=1+t;  y=3;      label=C1;} \ heated blade
  border C12(t=0,1){ x=2;    y=3-6*t;  label=C1;}
  border C13(t=0,1){ x=2-t;  y=-3;     label=C1;}
  border C14(t=0,1){ x=1;    y=-3+6*t; label=C1;}

  border C21(t=0,1){ x=-2+t; y=3;      label=C2;} \ cooling blade
  border C22(t=0,1){ x=-1;   y=3-6*t;  label=C2;}
  border C23(t=0,1){ x=-1-t; y=-3;     label=C2;}
  border C24(t=0,1){ x=-2;   y=-3+6*t; label=C2;}

            plot(    C00(50)
                   + C11( 5) + C12( 20) + C13( 5) + C14( 20)
                   + C21(-5) + C22(-20) + C23(-5) + C24(-20), wait=true);

  Th = buildmesh(    C00(50)
                   + C11( 5) + C12( 20) + C13( 5) + C14( 20)
                   + C21(-5) + C22(-20) + C23(-5) + C24(-20));

  plot(Th,wait=true);

  savemesh(Th, "example22.msh");
```

Example 22 mesh

C1 hot: boundary condition, **C2** is part of mesh



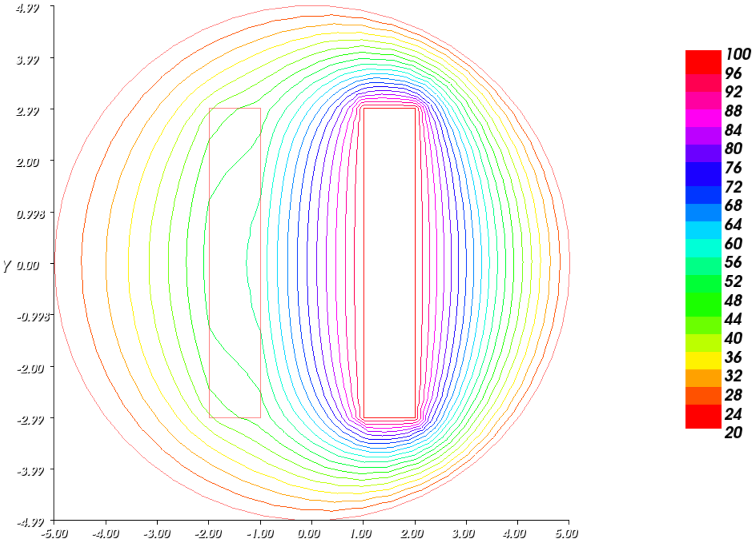
example22.edp code, cont'd

```
} else {
  Th = readmesh("example22.msh");
  C0= 99; C1= 98; C2= 97; // Numbers are in file, not labels
}
```

example22.edp code, cont'd

```
} else {
  Th = readmesh("example22.msh");
  C0= 99; C1= 98; C2= 97; // Numbers are in file, not labels
}

fespace Vh(Th,P1);
Vh u,v;

Vh kappa = 1 + 4*(x<-1) * (x>-2) * (y<3) * (y>-3);

solve a(u,v)= int2d(Th)( kappa*( dx(u)*dx(v) + dy(u)*dy(v) ) )
                + on(C0, u=20) + on(C1, u=100);

plot(u,value=true,wait=true,fill=false);
```

Example 22 results



Exercise 24 (10 points)

Example 22 is clearly symmetric about the x-axis. Modify the example so that it only solves *half* the problem, with a symmetry boundary (homogeneous Neumann condition) on the x-axis. Check your work visually by comparing the solutions. Pay particular attention to level curves that pass through the cooling blade (C2).