Re-examination of a result of Butler

In 2006, Ms. Trisha Butler completed a Master's thesis with Prof. W. Layton as advisor. Her thesis is available from the library (ETD) in pdf form.

The thrust of the thesis is a comparison of the differing rates of convergence of the *solution* of a PDE and of some *linear functionals* of the solution. To this end, she generated a two-dimensional numerical model of the advection-diffusion equation

$$
\begin{aligned}
-\varepsilon \Delta U + b \cdot \nabla U &= f(x, y) \text{ in } \Omega = [0, 1]^2 \\
U &= g \text{ on } \partial\Omega
\end{aligned}
$$

and applied the Jacobi iterative method to solve it. The linear functionals she examined are

1. The average U, given by $L(U) = \int_\Omega U(x, y)dxdy$, and
2. The flux of the solution at $x = 1$, given by $L_F(U) = \int_\Gamma \varepsilon \nabla U \cdot \hat{n} dx$.

For her work, she discretized the PDE using a finite difference method. Her thesis includes copies of the Matlab code she used.

In her conclusion, she remarked:

> The robustness of the post processing and descent algorithms were explored by augmenting these algorithms with the jacobi method to solve for the average and flux functionals. The data showed that the usefulness of the algorithms depended largely on which functional wa being calculated and in some cases, whether the size N of the matrix U was even or odd. For example, when the post processing algorithm was used with jacobi to calculate the average functional for the symmetric case and N was even, the functional L converged between 4-5 times faster than thee convergence of U. In contrast, when the post processing algorithm was used with Jacobi to calculate the flux functional for the symmetric case and N was odd, the functional L converged only 1.2-1.5 times faster than the convergence of U. Further, when any algorithm was used to solve for the flux functional in the nonsymmetric case, the rate of convergence to L decreased as the meshwidth h decreased.

For this project, you will perform a similar study, but using a finite element discretization of the problem. You will implement the Jacobi iterative algorithm (described fully in the thesis) and compare convergence rates of the solution and of the average functional for a few different matrix sizes. Your objective is to examine (at least) the cases described in Tables 1 and 2 in the thesis to determine if there remains a dramatic difference in convergence rates between even and odd values of N when using finite elements. You do not have to consider the largest of the problems if your computer resources are not sufficient.

As a second part of the project, repeat the study, replacing the unit square with Ω equal to the unit ball.