Natural Inclusion of Boundary Points in a Mesh

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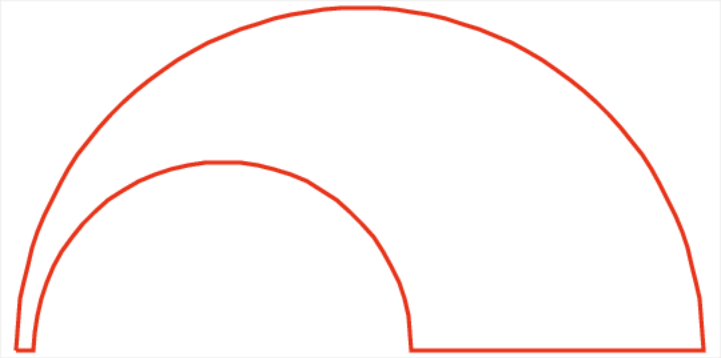
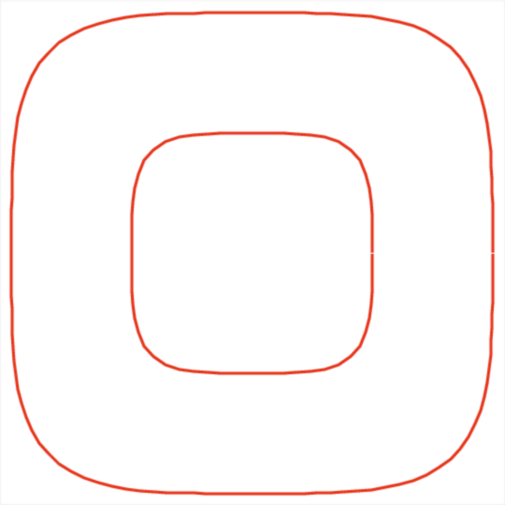
Meshes of Boundary and Interior

Given a region and its boundary, ...select a relatively coarse mesh of points ...as *automatically* as possible, ...with good distribution in the interior ...**and** on the boundary.

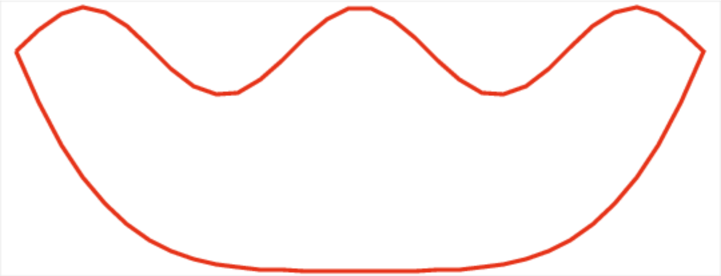
A Region and its Boundary: The Horn



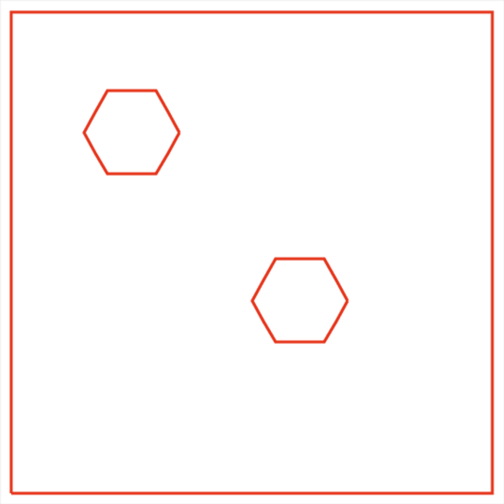
A Region and its Boundary: The Superellipse



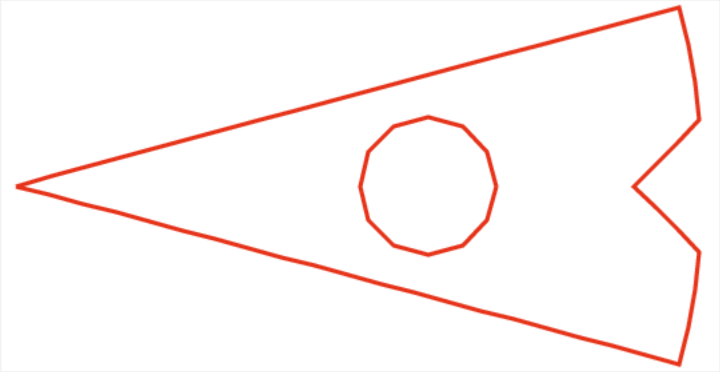
A Region and its Boundary: The Bicycle Seat



A Region and its Boundary: Two Hexagonal Holes



A Region and its Boundary: The "Holey" Pie



In a coarse mesh, uniform distribution can be difficult.

Forcing some points to lie on the boundary makes this harder.

Possible complicating factor: *Nonuniform density* .

Possible complicating factor: *data is 3D...or higher*.

CVT: Centroidal Voronoi Tessellation

We are interested in using CVT's for meshing and sampling.
An iteration is necessary to approximate the generators of a CVT:

- **Initialize** generators;
- **Sample** the region with thousands of points;
- **Estimate** Voronoi: assign samples to nearest generator.
- **Replace** each generator by centroid of Voronoi region;
- **Repeat** until convergence.

CVT Iteration: Initial Generators

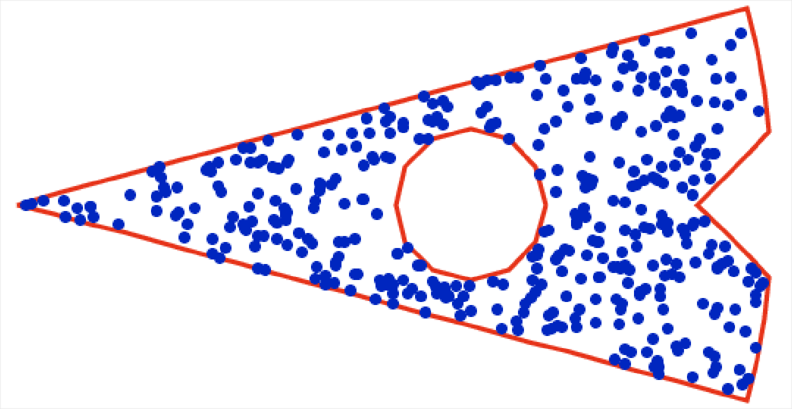


Figure: Generators Initialized Randomly

CVT Iteration: Final Generators

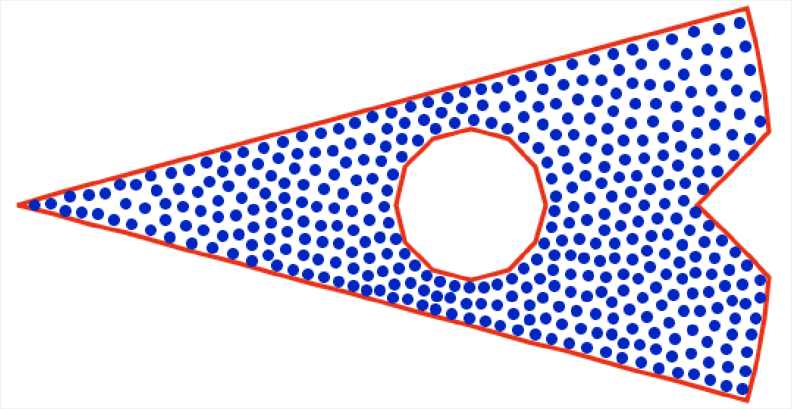
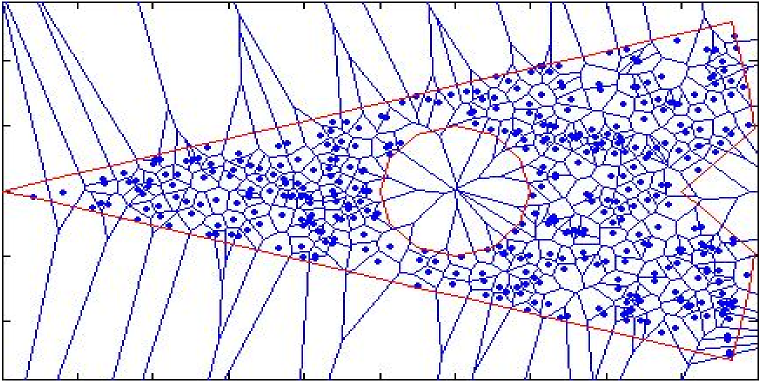


Figure: Generators after 40 Iterations

The boundary is not sampled. In fact it is avoided!

CVT Iteration: Animation



Can the CVT Algorithm be Adjusted?

We could move all nearby points onto the boundary. But that would disrupt the distribution and shape of the mesh.

We could fix a given number of points on the boundary, allowing them to slide. But how many points? Special problems with nonuniform densities.

How Shewchuk's TRIANGLE Gets the Boundary

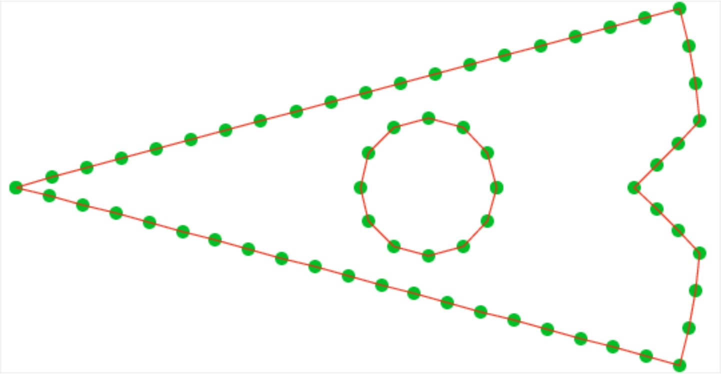


Figure: TRIANGLE requires "guide" nodes on the boundary

TRIANGLE works inward from Boundary Nodes

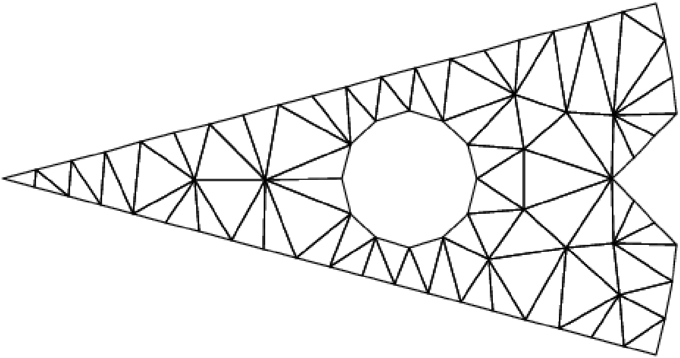


Figure: Mesh on boundary may be refined at points

A Finer TRIANGLE Mesh

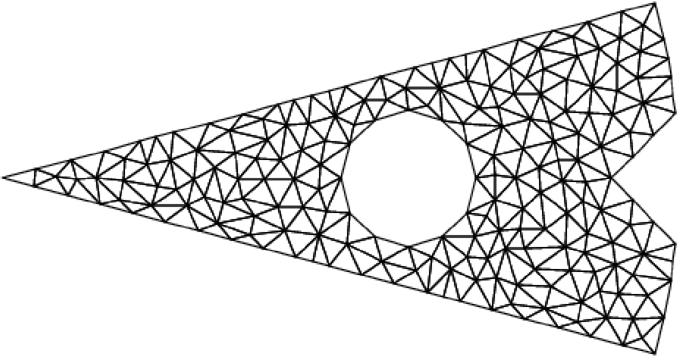


Figure: Triangles vary in shape and size

TRIANGLE's Mesh Generators May be Uneven

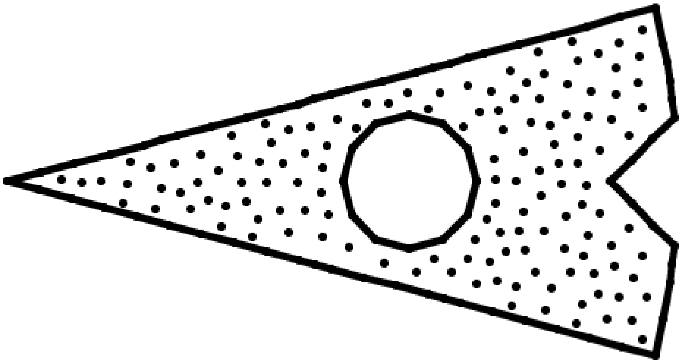


Figure: The points do not seem uniformly distributed

Persson and Strang's DISTMESH

- Compute a good mesh in the interior.
- Let it expand "a little" beyond the boundary.
- Push the exterior points back onto the boundary.
- Iterate this process so that it smooths out.

Initial DISTMESH Nodes

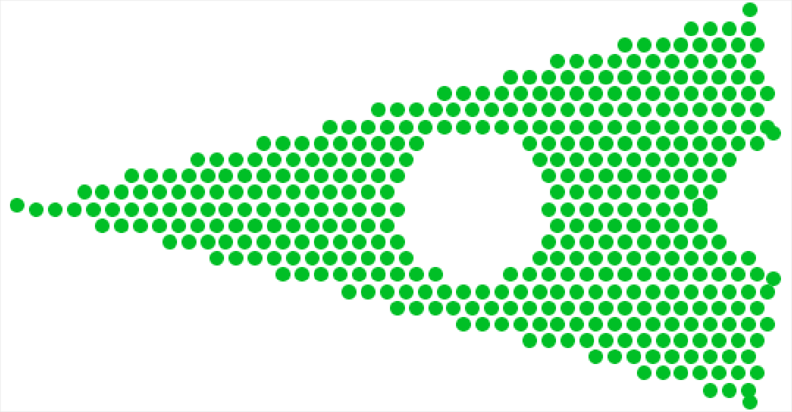


Figure: Hexagonal Lattice Data

Final DISTMESH Nodes

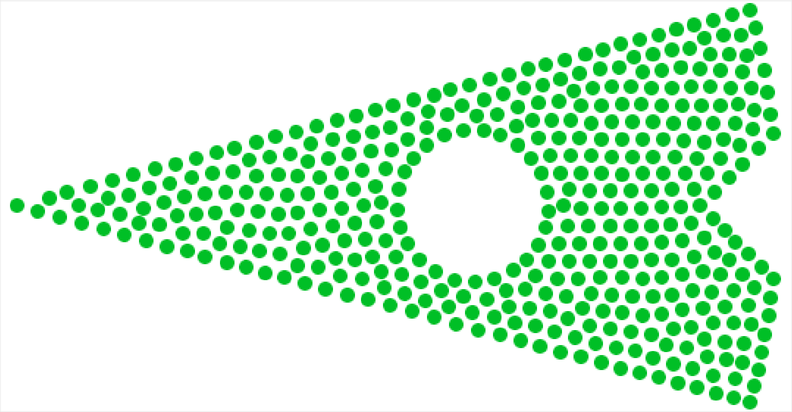


Figure: The nodes are uniformly distributed

Final DISTMESH Nodes and Mesh

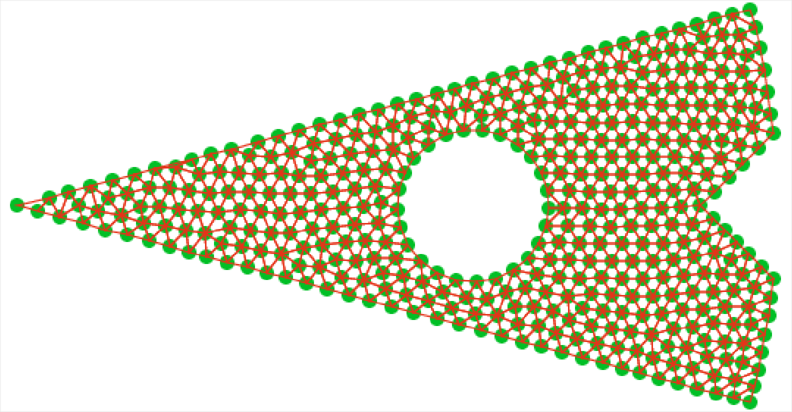


Figure: The derived mesh looks good

Virtues of DISTMESH Approach

- All points are treated equally;
- Boundary populated by “interested” interior points, so boundary shares density properties of interior;
- Boundary points are free to move or return to interior;

Adapt DISTMESH Ideas to CVT: Initialization

- Initialize generators as points on a hexagonal grid.

Hex Grid Initialization

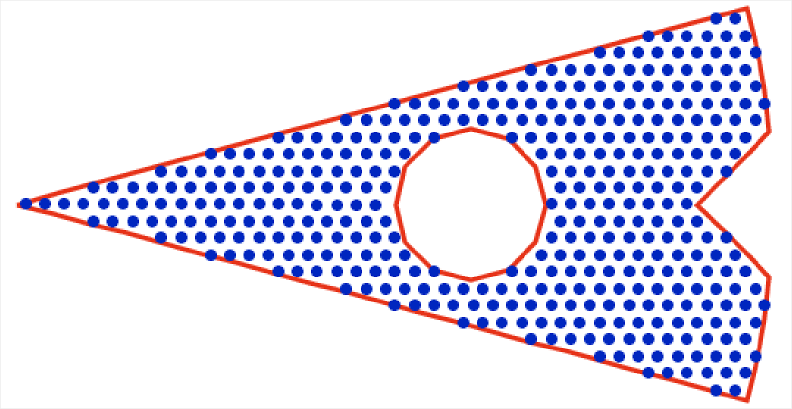


Figure: Initial points are on hexagonal grid

Smoother Results From Hex Initialization

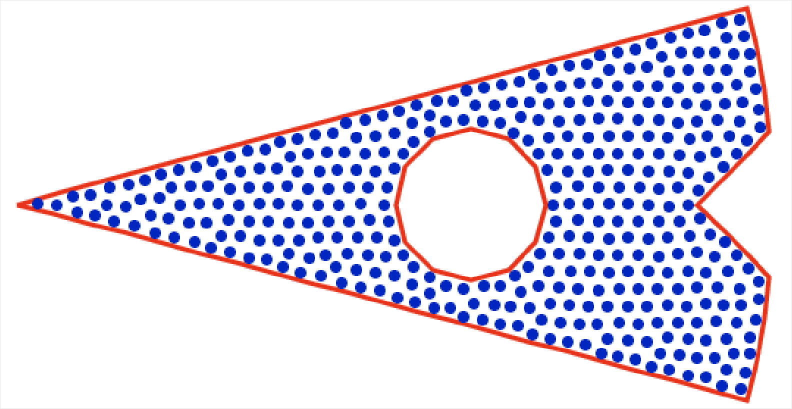


Figure: Generators after 40 Iterations

Adapt DISTMESH Ideas to CVT: Sampling

Need to draw CVT generators towards (and past!) the boundary.

- Sample a "fattened" region.
- Points outside the region will tend to pull centroids outside.

CVT samples a thickened region

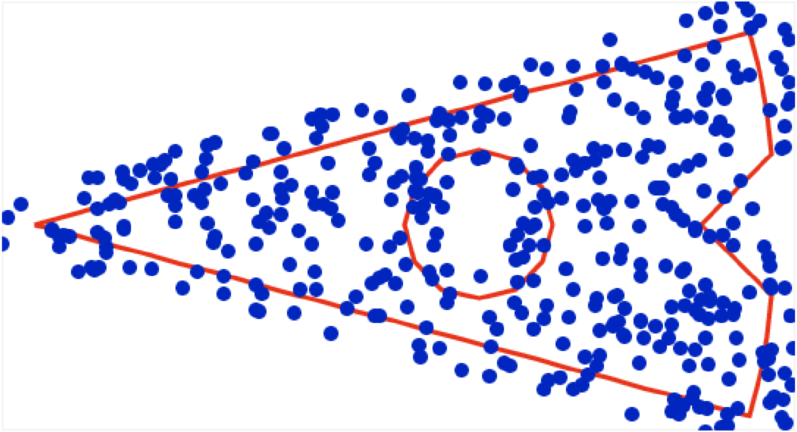


Figure: 400 samples

Adapt DISTMESH Ideas to CVT: Push Back to Boundary

- Determine whether a centroid is outside the region;
- Project such centroids back to nearest boundary point.

CVT generators using thickened region

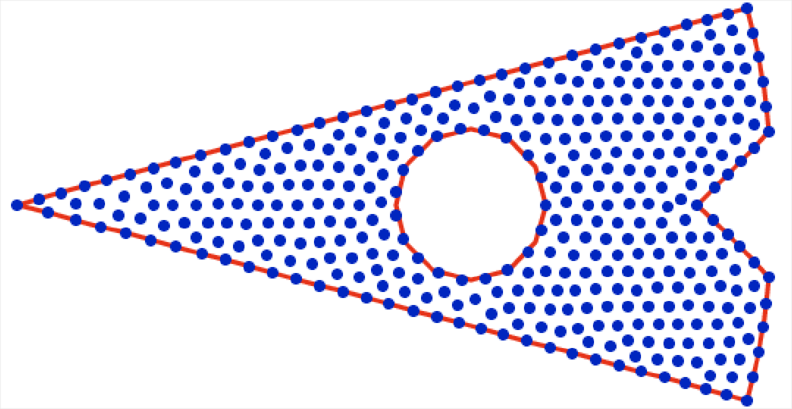


Figure: The converged generators

CVT Iteration with Boundary Capture

Initial Generators and associated Voronoi Diagram:

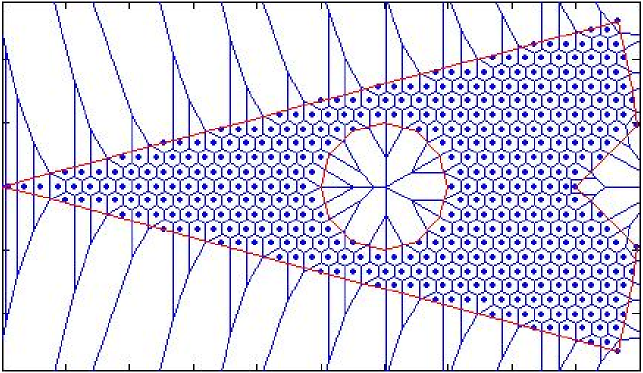


Figure: Initial "Hex" Data

CVT Iteration with Boundary Capture

Final Generators and associated Voronoi Diagram:

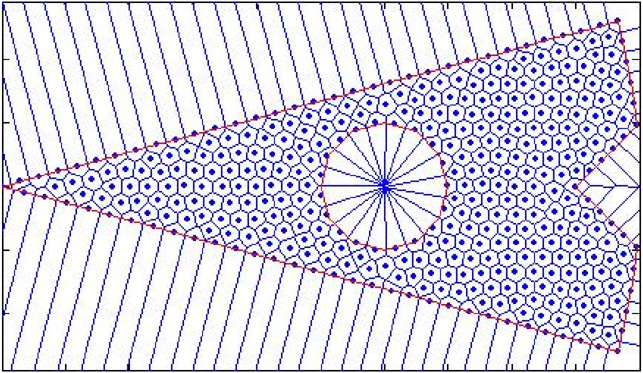
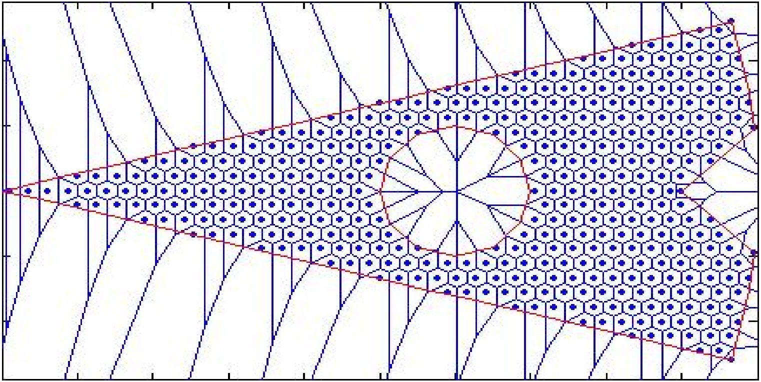


Figure: Final "CVT" Data

CVT Iteration with Boundary Capture: Animation



The "Hexnut"

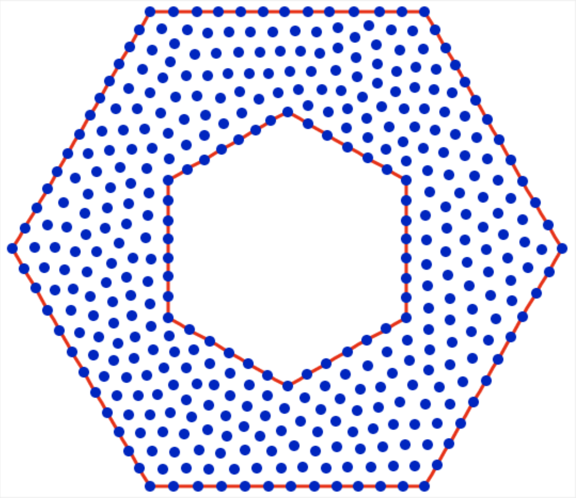


Figure: The converged generators

The "Bicycle Seat"

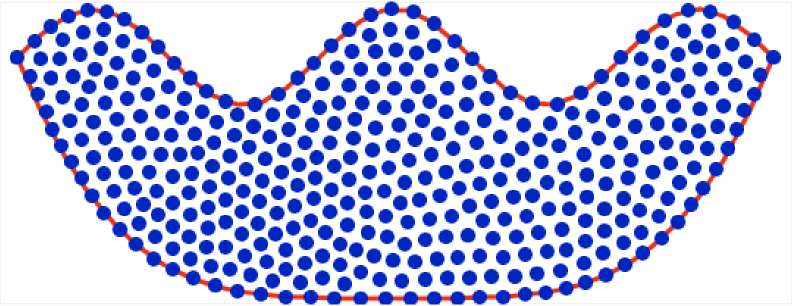


Figure: The converged generators

Nonuniform Density

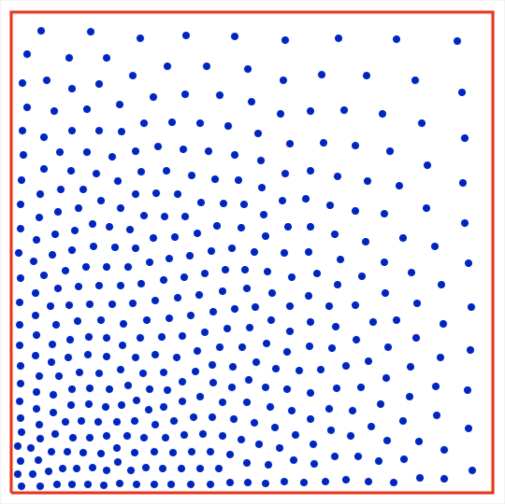


Figure: Computation with No Boundary Capture

Nonuniform Density

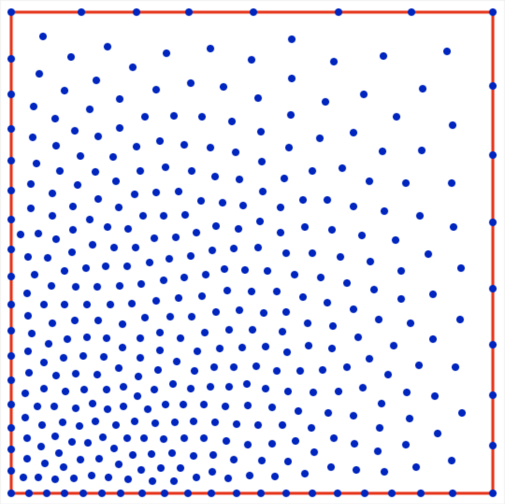


Figure: Boundary Capture

Issues

- Need to formulate objective functions;
- Need to formulate quality measures;
- Need a way to choose extra region thickness;
- Perhaps a better way to push generators towards boundary;

Future

- Problems in 3D can be handled in the same way;
- Nonuniform mesh density functions can be used;
- Voronoi regions could be computed exactly;
- Investigate relationship with Persson/Strang algorithm;

More Information

- Qiang Du, Vance Faber, Max Gunzburger, Centroidal Voronoi Tessellations, SIAM Review, December 1999.
- Per-Olof Persson and Gilbert Strang, A Simple Mesh Generator in MATLAB, SIAM Review, June 2004.
- Jonathan Shewchuk, Delaunay Refinement Algorithms for Triangular Mesh Generation, Computational Geometry, May 2002.