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## PART III - CRYPTOGRAPHY

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You may know what this means . . .



“Fascinating – I used to work on codes myself, during the war.”

But do you know what this means?

GHIHQG WJH HDVW ZDOO RI WKH FDVWOH

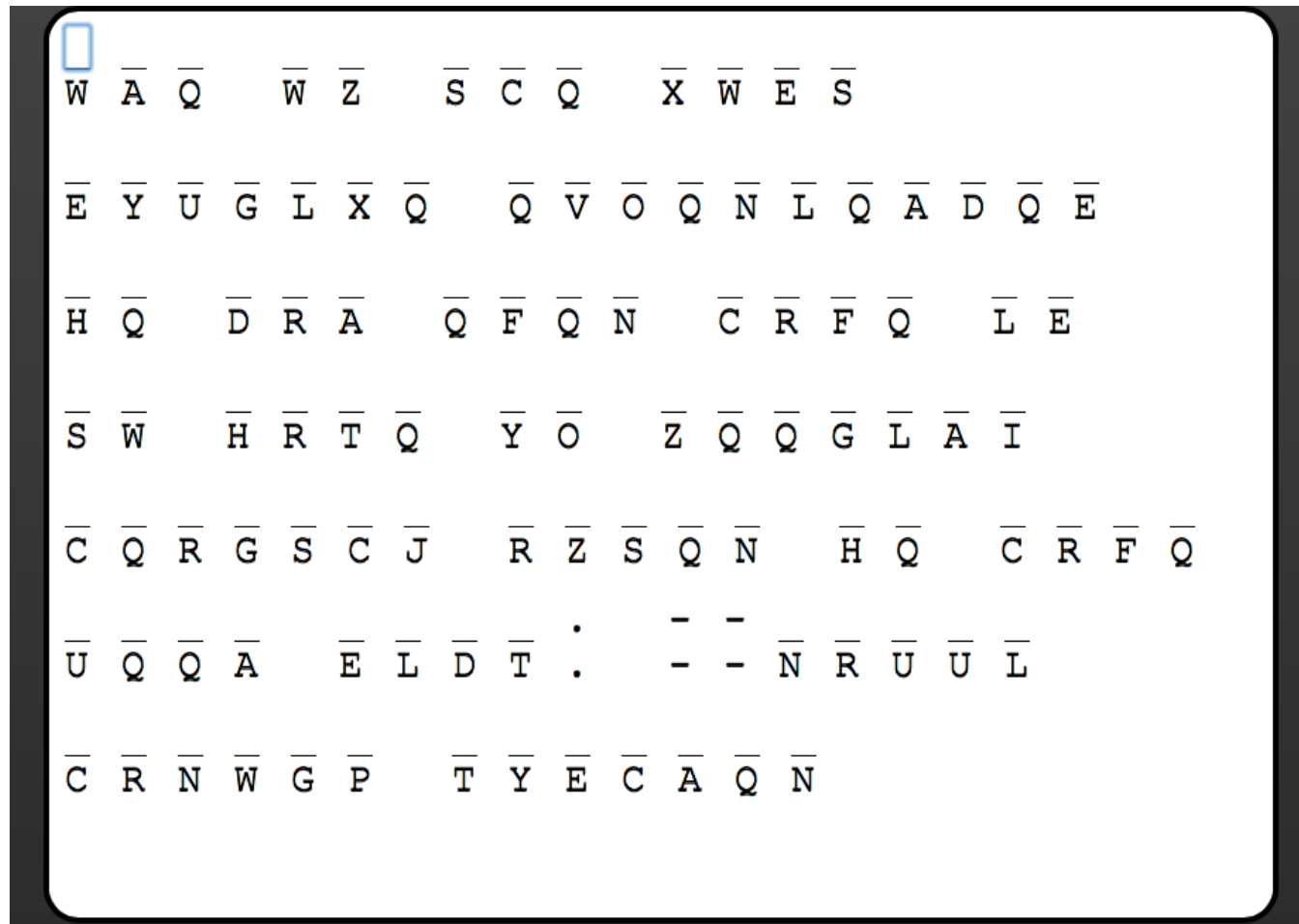
Julius Caesar (100 BC - 44BC) could have sent this message! Because he didn't trust his generals, when he sent messages he did something like replace every "A" with a "D", every "B" with an "E", every "C" with an "F", etc. See the pattern? We call this the **key** which allows us to decode the message. It is called the **Caesar Cipher**.

GHIHQG WJH HDVW ZDOO RI WKH FDVWOH

G	H	I	H	Q	G	W	J	H	H	D	V	W
↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓
D	E	F	E	N	D	T	H	E	E	A	S	T

DEFEND THE EAST WALL OF THE CASTLE

Have you downloaded an app for Cryptograms or seen a puzzle like this in a magazine or newspaper?



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## Goals for Part III

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1. To learn basic terminology of cryptography
2. To see how to code and decode simple ciphers if the key is given
3. To see how to decode simple substitution ciphers without a key using frequency of letters and words
- 4.

## Goals for this lecture:

1. See how ciphers have been used in the past.
2. Learn basic terminology
3. Learn how to write a message using the Caesar cipher
4. Learn how to decode a message using the Caesar cipher
5. To understand why this type of cipher is not secure

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## Atbash Cipher

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An even earlier code (circa 500 BC) than the Caesar cipher was used for the Hebrew alphabet. Instead of shifting a letter to the right it replaces the first letter of the Hebrew alphabet (aleph) with the last letter (tav); then the second letter (beth) for the second from last letter (shin), etc. So it uses a substitution where the letters are reflected.

### The ATBASH Cipher

א ב ג ד ה ו ז ח ט י כ ל ם נ ס ע פ צ ק ר ש ת  
ת ש ד ק צ פ ע ס נ ם ל כ י ט ח ו ז ה ד ג ב א

If we used an Atbash-like cipher with our alphabet we would make a substitution like the following.

A B C D E F G H I J K L M N O P Q R S T U V W X Y Z

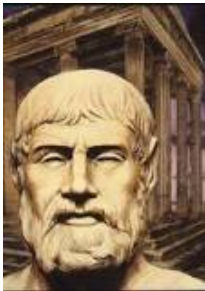
Z Y X W V U T S R Q P O N M L K J I H G F E D C B A



## Other Early Attempts at Secrecy

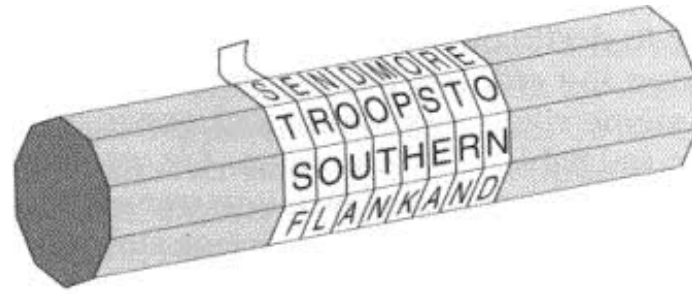
Early attempts were often messy. One strategy was to hide it on the messenger.

- The Chinese wrote messages on silk and encased them in a ball of wax which would then be hidden *in* the messenger.
- The Greeks (around 500 BC) used a different approach. Histaeus wanted to send a message to Aristogorus to urge revolt against the Persians but he needed secrecy. He shaved the head of his most trusted slave and tattooed the message on his head. When the hair grew back the slave was sent to Aristogorus who shaved his head and read the message. I guess they weren't in a hurry!





The Spartans used a tool called a scytale.



A piece of parchment is wound around a baton and a message is written on it. Then the parchment is removed and sent to the recipient. Of course the recipient must have the same size baton for it to work.

How could you break this code?

There were no significant improvements in creating more complex ciphers until the middle ages.

Leon Battista Alberti, a Venetian, created something called a Cipher disk and described how to use it in a 1467 treatise. One disk is fixed and you turn the other.



You turn the disk and the letters that match up from one disk to another give the substitution. Note that the letters are not in order in the center disk. In this picture the substitution V for A, Z for B, etc. is used.

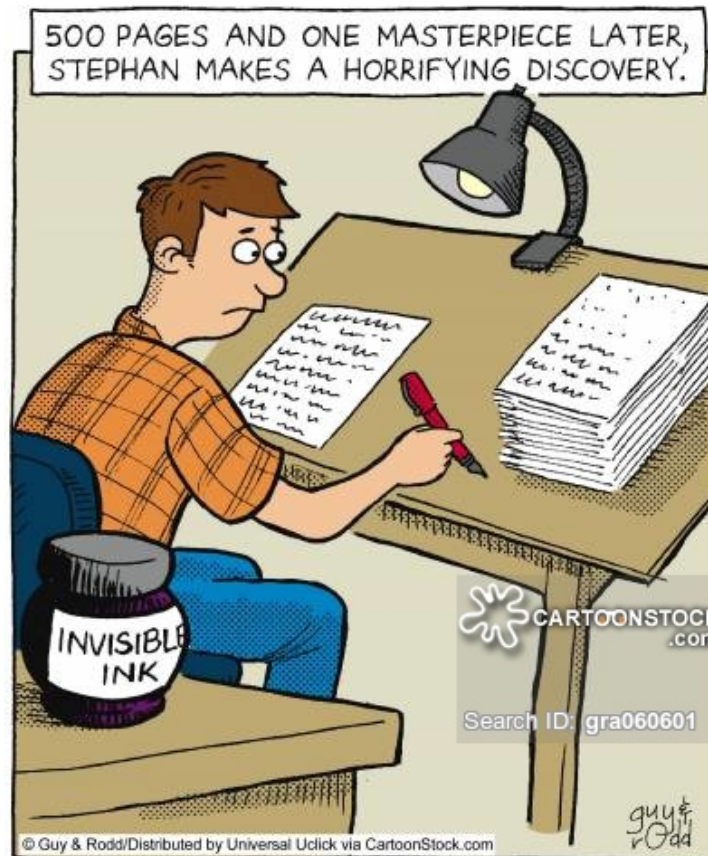
## Invisible Ink

Using invisible ink is a less intrusive method of guaranteeing secrecy. The simplest choices are organic compounds such as lemon juice, milk, or urine. During the American revolution both sides made extensive use of chemical inks that required special developers. It was used up through World War I.



An invisible ink letter treated with a chemical reagent from British spy Benjamin Thompson, 1775 (Clements Library, University of Michigan).

There is a book entitled **Invisible Ink: Spycraft of the American Revolution** by John Nagy which you might find interesting.



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## Encryption & Decryption

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- We call **PLAINTEXT** data that can be read and understood without any special measures. Examples: these notes, books, magazines, etc.
- **ENCRYPTION** is the method of disguising plaintext so that the information is hidden.
- **CIPHERTEXT** is encrypted plaintext. Example: the encrypted message on the second slide using the Caesar Cipher.
- **DECRYPTION** is the process of changing ciphertext to plaintext.



PLAINTEXT

CIPHERTEXT

PLAINTEXT

ENCRYPTION

DECRYPTION

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## What is Cryptography?

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- Cryptography is the study of using methods to **encrypt** information or **decrypt** ciphertext.
- In the past, cryptography has been used to protect secrets such as military information.
- Now we use methods to encrypt sensitive data such as personal information and to transmit information across insecure networks so it can only be read by the intended recipient.
- In the far past, people used pen & pencil to encrypt messages but now mathematical algorithms are used.
- Cryptography is everywhere now.
  - Web traffic
  - Wireless traffic like blue Tooth

- Encrypting files on your computer disk
  - Content protection (DVDs, Blu-ray, etc.)
  - Digital signatures
  - etc.
- We will first look at simple encryption methods which can be used to encrypt or decrypt messages by hand.
  - Then we will investigate the ideas behind sophisticated encryption methods that are used in your daily life.

Here is a quote that is worthwhile to keep in mind.

“There are two kinds of cryptography in this world: cryptography that will stop your kid sister from reading your files and cryptography that will stop major governments from reading your files.”

– Bruce Schneier



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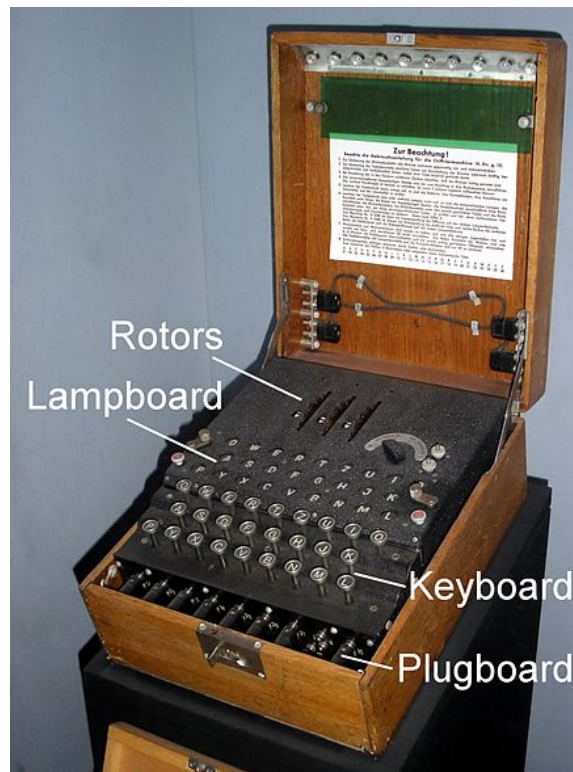
## Codebreaking in World War II

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Did you see the movie *The Imitation Game* or read a book about *Alan Turing* such as the *Enigma Machine*?



- After WWI the Germans built a machine called **the enigma machine** to protect diplomatic and military communications.



- An operator typed the text of the communication on the keyboard and the machine encrypted the message.
- Three Polish cryptologists (working for Polish military) developed a mechanical

device for breaking the Enigma ciphers but the Germans then added additional complexity to the machines to make decryption much more difficult.

- Cryptographers during WWII (led by Alan Turing) devised an electromechanical machine (a forerunner of a computer) which found the **key** to encrypt the messages and thus they could decrypt the messages.
- This played a pivotal role in reading encrypted German messages that enabled the Allies to defeat the Nazis in many crucial engagements.
- For more information on the Enigma machine go to [cryptomuseum.com](https://cryptomuseum.com) There is an app there you can download to simulate the Enigma machine.

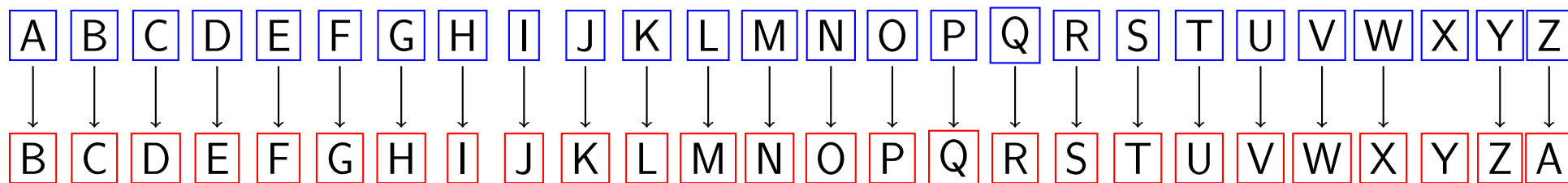
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## Shifted Ciphers (Caesar Cipher)

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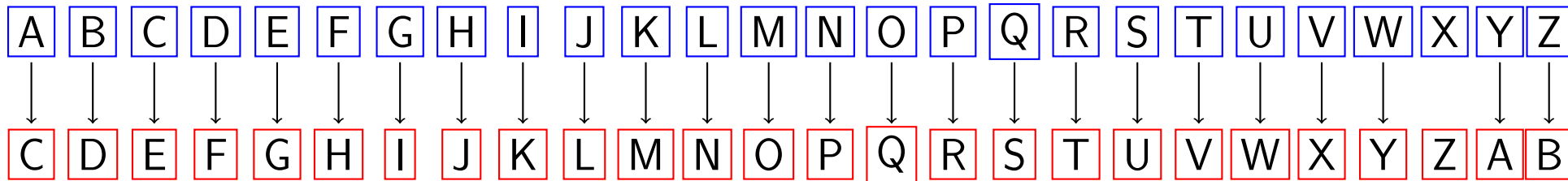
- The example we showed on the first slide used the Caesar cipher where each letter of the alphabet was replaced by the letter which was three letters after it (in a circular fashion).
- However, we can use any number of letters between 1 and 25 to shift the letter.
- This means that **there are only 25 possible ways to decrypt any message using this simple shift method.**

Shift 1 letter to right

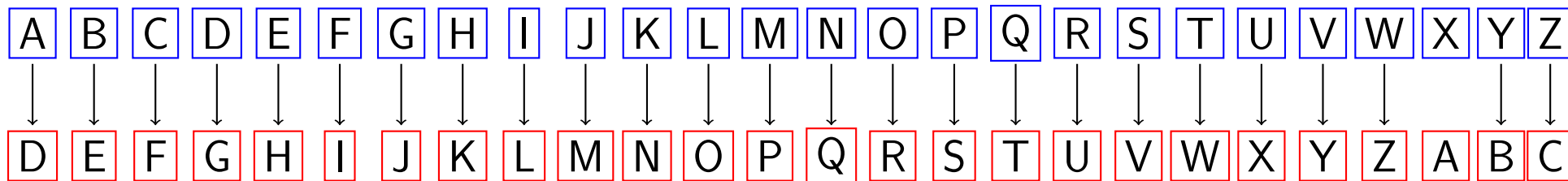


“Z” is replaced by “A” in this cipher because we shift the alphabet in a cyclic manner.

Shift 2 letters to right



Shift 3 letters to right



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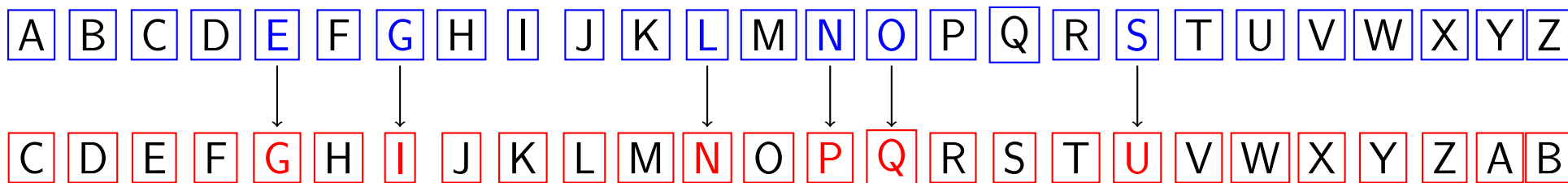
**Example.** Encrypt the message

GO NOLES

using a shifted cipher where you shift by 2 letters.

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We first write the letters of the alphabet and then under it, the letters of the alphabet shifted by two letters



To do this we take a “G” from the top line and see that below it is an “I”, below the “O” is a ” Q”, etc.

IQ PQNGU

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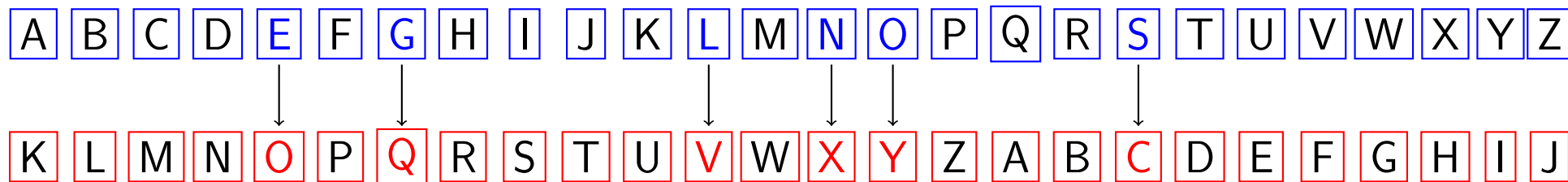
**Example.** Encrypt the message

GO NOLES

using a shifted cipher where you shift by 10 letters.

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We first write the letters of the alphabet and then under it, the letters of the alphabet shifted by ten letters.



QY XYVOC

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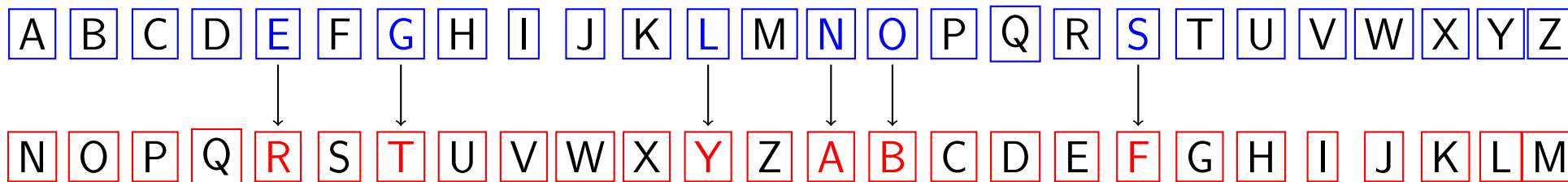
**Example.** Encrypt the message

GO NOLES

using a shifted cipher where you shift by 13 letters.

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We first write the letters of the alphabet and then under it, the letters of the alphabet shifted by ten letters.



TB ABYRF

Note that because we have shifted by 13 characters and our alphabet has 26 characters ( $2 \times 13$ ) it is called a **symmetric cipher** and given the name **ROT13**. This is because to encrypt and decrypt you use the same key. For example,  $A \rightarrow N$  and  $N \rightarrow A$ . ROT13 is used in online forums as a means of hiding spoilers, punchlines, puzzle solutions,



and offensive materials from the casual glance. It is the basis for many puzzles and games and is thought to be the equivalent of a magazine printing the answers to a puzzle upside down.

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## SOCRATIVE QUIZ - PartIII\_Practice\_Quiz1

IUZGAZ34E

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1. Encrypt FSU using a 5 letter shift.
2. Encrypt FSU using a 6 letter shift.

HINT for # 1

A B C D E F G H I J K L M N O P Q R S T U V W X Y Z

F G H I J K L M N O P Q R S T U V W X Y Z A B C D E

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## Decrypting a Shifted Cipher

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Decrypting a shift cipher is harder than encrypting because we have to go backwards! To encrypt all we need is to follow the key and do it once but with decrypting a shift cipher we have to consider possibly all choices. Lucky for us for a shifted cipher there are only **25 possibilities**.

Consider the encrypted message **QY XYVOC**

If this had been encrypted with a 1-letter shift then we write the letter before “Q” (which is “P”), then the letter before “Y” which is “X”, etc. to get **PX WXUNB** which clearly isn’t decrypted. So the message wasn’t encrypted using a 1-letter shift.

If the original message was formed using a 2-letter shift then we write the letter two before “Q”, etc. or more simply, one letter before the 1-letter shift **PX WXUNB** to get **OW VWUMA**.

Continuing in this manner, we get the following table and see that the message was encrypted with a 10 letter shift.

A B C D E F G H I J K L M N O P Q R S T U V W X Y Z

Shift 0 QY XYVOC  
Shift 1 PX WXUNB  
Shift 2 OW VWUMA  
Shift 3 NV UVT LZ  
Shift 4 MU TUSKY  
Shift 5 LT STRJX  
Shift 6 KS RSQIW  
Shift 7 JR QRPHV  
Shift 8 IQ PQNGU  
Shift 9 HP OPMFT  
Shift 10 GO NOLES  
Shift 11 FN MNKDR

We have to do at most 25 of these shifts to find the answer. Easy for us to do and even easier for a computer!

Because we have to do at most 25 shifts it doesn't really matter whether we shift backwards or forwards. So for the previous problem we could decrypt it by shifting to the right.

QY XYVOC  $\implies$  RZ YZWPD  $\implies$  SA ZAXQE  
 $\implies$  TB ABYRF  $\implies$  UC BCZSG  $\implies$  VD CDATH  
 $\implies$  WE DEBUI  $\implies$  XF EFCVJ  $\implies$  YG FGDWK  
 $\implies$  ZH GHEXL  $\implies$  AI HIFYM  $\implies$  BJ IJGZN  
 $\implies$  CK JKHAO  $\implies$  DL KLIBP  $\implies$  EM LMJCQ  
 $\implies$  FN MNKDR  $\implies$  GO NOLES

In this example it was shorter to shift backwards but this is not the case in general as the following decryption illustrates.

Backward Shift	Decryption
0	LMTK PTKL
1	KLSJ OSJK
2	JKRI NRIJ
3	IJQH MQHI
4	HIPG LPGH
5	GHOF KOFG
6	FGNE JNEF
7	EFMD IMDE
8	DELC HLCD
9	CDKB GKBC
10	BCJA FJAB
11	ABIZ EIZA
12	ZAHY DHYZ
13	YZGX CGXY
14	XYFW BFWX
15	WXEV AEVW
16	VWDU ZDUV
17	UVCT YCTU
18	TUBS XBST
19	STAR WARS

Forward Shift	Decryption
0	LMTK PTKL
1	MNUL QULM
2	NOVM RVMN
3	OPWN SWNO
4	PQXO TXOP
5	QRYP UYPQ
6	RSZQ VZQR
7	STAR WARS

Typically in a shift cipher the key is given as a shift to the right. So to describe the key used in this cipher, we say that we shift the alphabet 19 letters to the right. Equivalently, we could describe it as 7 letters to the left.

A B C D E F G H I J K L M N O P Q R S T U V W X Y Z

T U V W X Y Z A B C D E F G H I J K L M N O P Q R S

It's a bit tedious to go through all of these shifts every time. In fact, this is the Brute Force approach for decrypting a shifted cipher. Is there anything we can do to speed up the process?

1. First we write down the alphabet across the top of a square. Then in a second row we write the alphabet shifted by one letter, in the third row the alphabet shifted by 2 letters, etc. It is called a **Vigenere square**.
2. Next we use some things we know about the English language. First, **E** is the most common letter followed by **T**. Also we can take a short word and see which shift makes sense. We will explore more about the frequencies of letters and words in the English language in the next lecture.
3. This Vigenere square makes encrypting messages with a shifted cipher easy too.



	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
S1	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A
S2	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B
S3	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C
S4	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D
S5	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E
S6	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F
S7	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G
S8	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H
S9	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I
S10	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J
S11	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K
S12	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L
S13	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M
S14	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N
S15	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
S16	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P
S17	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q
S18	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R
S19	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S
S20	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T
S21	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U
S22	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V
S23	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W
S24	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X
S25	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y

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**Example.** Use the fact that "E" is the most frequently appearing letter in the alphabet and "T" the second most to decode the message

XJINODOPODJI

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- The letter "O" appears 3 times in this expression so we first see if this represents "E". So we look in the column with "E" at the top and go down the column until we see an "O". This occurs in Shift 10 the letter "O" represents "E" and so the decrypted message is NZ ... which doesn't make sense.
- Next we try to see if "O" represents "T". We look at the column with "T" at the top and go down the column until we find an "O". If we look at the row beginning with "V" we see that this is the case so we apply Shift 22 to get CONSTITUTION

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
S1	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A
S2	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B
S3	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C
S4	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D
S5	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E
S6	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F
S7	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G
S8	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H
S9	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I
S10	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J
S11	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K
S12	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L
S13	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M
S14	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N
S15	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
S16	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P
S17	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q
S18	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R
S19	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S
S20	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T
S21	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U
S22	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V
S23	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W
S24	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X
S25	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y

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**Example.** Use the fact that "E" is the most frequently appearing letter in the alphabet, "T" the second most and "A" the third to decode the message

IZUILQTTW

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- The letter "I" appears 2 times in the encrypted message and the letter "T" appears two times (consecutive).
- We first see if "I" represents "E". We look at the top row and find the column with "E" and then go down the column until we find an "I" which occurs in Shift 4 but the decryption would begin **EVQEH** which doesn't make sense.
- Next we try to see if "I" represents "T". If we look at the column with "T" at the top and go down the column until we find an "I" we see that this occurs in Shift 15. In this case we get **TKFTW** which doesn't make sense.
- Next we try to see if "I" represents "A". If we look at the column beginning with "A" we see that using Shift 8 gives this representation and we get **ARMADILLO**

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
S1	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A
S2	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B
S3	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C
S4	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D
S5	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E
S6	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F
S7	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G
S8	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H
S9	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I
S10	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J
S11	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K
S12	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L
S13	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M
S14	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N
S15	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
S16	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P
S17	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q
S18	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R
S19	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S
S20	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T
S21	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U
S22	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V
S23	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W
S24	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X
S25	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y

---

Example. Decrypt the message

OZZ WG KSZZ

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- The letter “Z” appears 4 times but it probably doesn’t represent an “E” (we would have “EE” twice) or a “T” (we would have “?tt”) or an “A” so we have to use a different strategy.
- The first word has a double letter at the end which is probably a consonant so the first letter is probably a vowel. Let’s try this first.
- Looking at our square (see next slides with Vignere square highlighted), we now ask ourselves in which shifts does O represent a vowel. To do this we start with “O” at the top row and follow the diagonal consisting of “O” and see which shift gives a vowel. The first encountered is “I”, then “E”, etc. We omit “O” because this is the unshifted alphabet and we know shift 0 doesn’t make sense.
  - Shift 6 O  $\implies$  I
  - Shift 10 O  $\implies$  E
  - Shift 14 O  $\implies$  A

– Shift 20 O  $\implies$  U

- Now we look at what “Z” represents in each of these four shifts.

– Shift 6 OZZ  $\implies$  ITT

– Shift 10 OZZ  $\implies$  EPP

– Shift 14 OZZ  $\implies$  ALL

– Shift 20 OZZ  $\implies$  UFF

- The only one that makes sense is Shift 14 so the message is decrypted as

ALL IS WELL

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
S1	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A
S2	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B
S3	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C
S4	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D
S5	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E
S6	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F
S7	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G
S8	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H
S9	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I
S10	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J
S11	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K
S12	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L
S13	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M
S14	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N
S15	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
S16	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P
S17	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q
S18	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R
S19	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S
S20	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T
S21	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U
S22	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V
S23	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W
S24	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X
S25	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y



## We have seen two approaches for decrypting the Shift Cipher

1. **Brute Force.** In this approach we try to decrypt the message by shifting the alphabet by 1 letter and if this doesn't work we try a shift of 2 letters, etc. We know that we have to do **at most 25 shifts** to decrypt the message.
2. **Use our knowledge of the English language.** In this approach we use common knowledge about our language to make educated guesses for letters and see if a particular shift gives the decoded message. A Vignere square is helpful.

For a **Shift Cipher** it doesn't really matter which approach we choose because the Brute Force approach only takes a maximum of 25 tries. But for other ciphers we will see that the number of possibilities is so large that a brute force approach is not practical.

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## SOCRATIVE QUIZ - PartIII\_Quiz1

IUZGAZ34E

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1. \_\_\_\_\_ is the science and art of transforming messages to make them secure.
  - a. Cryptography
  - b. Encryption
  - c. Decryption
  - d. Caesar Cipher
  
2. \_\_\_\_\_ is the original message before encryption.
  - a. Key
  - b. Ciphertext
  - c. Plaintext
  - d. None of the above
  
3. \_\_\_\_\_ is the message after encryption.
  - a. Key

- b. Ciphertext
- c. Plaintext
- d. None of the above

4. \_\_\_\_\_ is the procedure that transforms plaintext to ciphertext.

- a. Key
- b. Encryption
- c. Decryption
- d. Cryptography

5. \_\_\_\_\_ is the procedure that transforms ciphertext to plaintext.

- a. Key
- b. Encryption
- c. Decryption
- d. Cryptography

6. In a Shift Cipher when the Key is **shift to right 4 letters**, how is the letter "P" encrypted?

- a. "R"

- b. "S"
- c. "T"
- d. "U"

7. In a Shift Cipher when the Key is **shift to LEFT 2 letters**, how is the letter "P" decrypted?

- a. "N"
- b. "R"
- c. "M"
- d. "S"

8. In a Brute Force approach to decrypting a Shift Cipher where 5 different letters are used in the message, the maximum number of tries is

- a. 6
- b. 5
- c. 4
- d. 3

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## Goals for this Lecture

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1. To understand what a monoalphabetic substitution cipher is.
2. To learn how to encrypt and decrypt a message using a simple substitution cipher if the key is given.
3. To understand that a “Brute Force” approach to decrypting a simple substitution cipher is not feasible.
4. To learn how to interpret and make frequency plots for the occurrence of letters and words in a given text.
5. To see how to use frequency analysis to decrypt a simple substitution cipher.
6. To understand the weaknesses in a monoalphabetic substitution cipher.

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## Simple Substitution Ciphers

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- In the examples in the previous lecture, the alphabet was **shifted** but the letters remained in the **same order**. This type of cipher is the simplest type of **monoalphabetic substitution cipher** which uses only **one** fixed alphabet.
- For example, when we used a Caesar Cipher with Shift 3 to right we used the single alphabet starting with “D” as a substitution alphabet

D E F G H I J K L M N O P Q R S T U V W X Y Z A B C

- Also the Atbash cipher is a monoalphabetic cipher but NOT a shifted cipher. Here we use the following alphabet as a substitution.

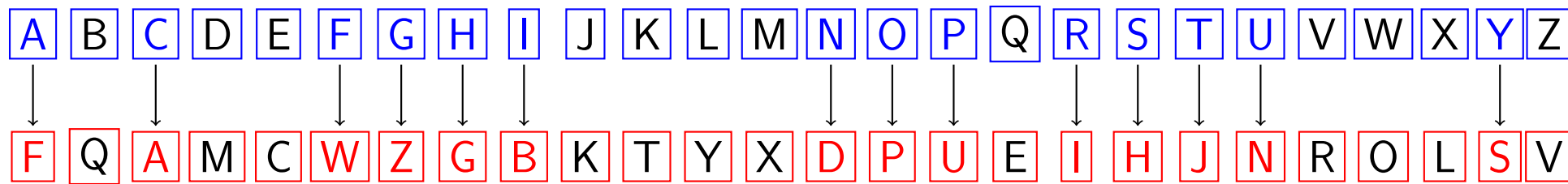
Z Y X W V U T S R Q P O N M L K J I H G F E D C B A

- We saw that it was easy to decrypt a message encrypted with a shifted (monoalphabetic) key because in English we have to do at most 25 shifted versions of the message to find the answer. We can also shorten our work by investigating the mostly commonly appearing letters.
- However, suppose we still just use the English alphabet but we mix up the order of the letters and then use this as a substitution, decrypting the message becomes a lot harder to do by hand. For example, suppose we use the key

A B C D E F G H I J K L M N O P Q R S T U V W X Y Z

B D F A C E H J T Z I X N M L K O V W U P G Y S Q R

Encrypting a message is always straightforward. For example, if we have the key



then the message

CRYPTOGRAPHY IS FUN

is encrypted as

AISJPZIFUGS BH WND



Decrypting a message is easy **if we have the key**. We just take the letter from the second line and write the corresponding letter on the upper line.

For example, given the Atbash key decrypt the message **HLXIZGREV**.

- We first find “H” on the second line (the key) and see that it represents “S”
- Next we find “L” on the second line and see that it represents “O”.
- Continuing in this manner we get **SOCRATIVE** as the decrypted answer.

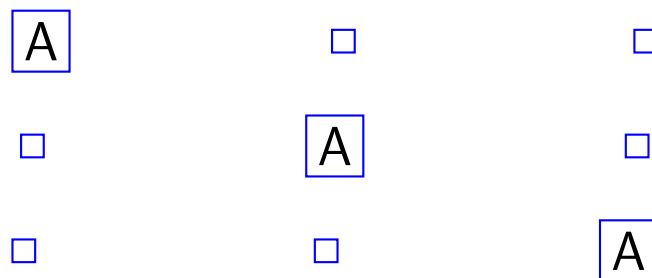
A B C D E F G H I J K L M N O P Q R S T U V W X Y Z

Z Y X W V U T S R Q P O N M L K J I H G F E D C B A

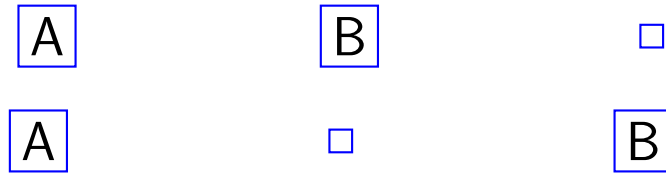
To decrypt a message (without a key) when a simple substitution cipher is used with the letters of the alphabet mixed up can we just use “Brute Force”, i.e., try all possibilities?

- Clearly we can't try all possible combinations of the alphabet by hand but is it reasonable to write a computer algorithm which would use the Brute Force method of trying all possible combinations as we did for the shifted encryption?
- The answer is **NO!**
- To see this we first look at a simplified alphabet.

First assume we have an alphabet of 3 letters, A, B, C. we see that there are 3 places to put the letter “A”

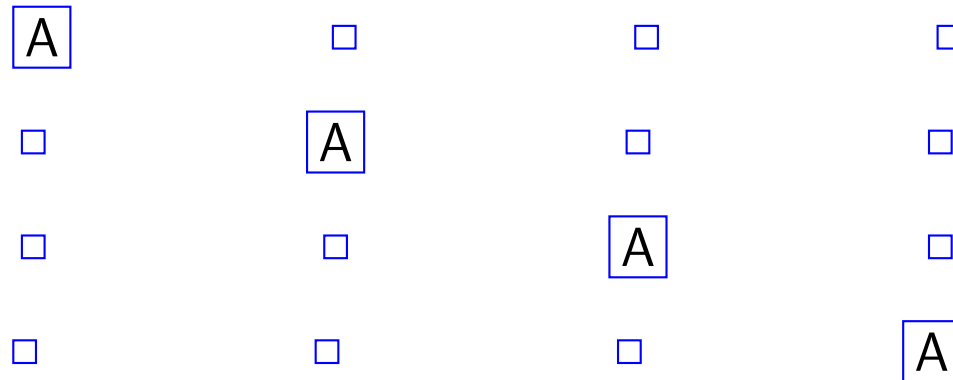


Now in each of the three cases there are two places to put the letter “B”. For example,

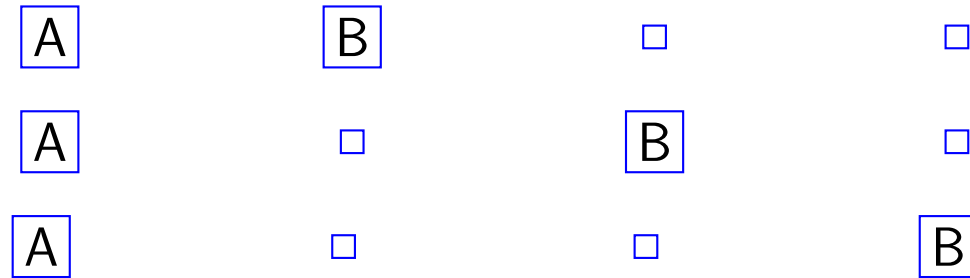


And since there are 3 locations for “A” and for each location there are 2 places to put “B” we have  $3 \times 2$  possibilities. Now in each of these six options there is only one place left for the letter “C” so we have  $3 \times 2 \times 1 = 6$  possibilities.

Now assume we have an alphabet of 4 letters, A, B, C, D and we want to see how the number of possible combinations increase. We see that there are 4 places to put the letter “A”



Now in each of the four cases there are three places to put the letter “B”. For example,



And since there are 4 locations for “A” and for each location there are 3 places to put “B” we have  $4 \times 3$  possibilities. Now in each of these 12 options there are two places left for the letter “C”. For example,



so we have  $4 \times 3 \times 2 = 24$  possibilities. Finally in each of the 24 options there is only one place left to put the letter “D” so we get  $4 \times 3 \times 2 \times 1 = 24$ . The number of possibilities has increased by a factor of 4.

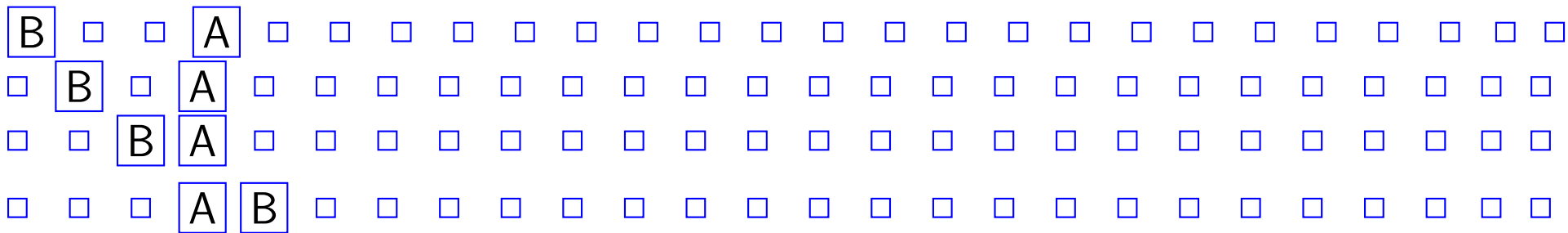
Now we want to do the same thing, except when we have 26 letters in the alphabet.

- First we see that there are 26 places to put the letter “A”

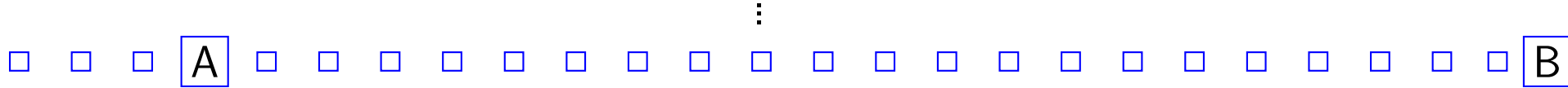


26 choices of where to put “A”

- After we have fixed the letter “A”, we fix the letter “B”. Now since “A” takes up 1 placeholder there are only 25 places where we can put the letter “B”. But there are 26 different places where we can put “A” and for each of these choices there are 25 places to put “B”. This means there are  $26 \times 25 = 650$  possibilities for where “A” and “B” can go.



⋮



Now there are  $26 \times 25$  possible combinations of “A” and “B”.

- Once we fix “A” and “B” there are 24 places left to put “C”. So for EACH of the 650 ( $26 \times 25$ ) possible combinations for “A” and “B” there are 24 possibilities of where to put “C”. So far we have  $26 \times 25 \times 24 = 15,600$  possible combinations.



- After we set the locations of “A”, “B”, and “C” there are 23 locations left to place “D”. This means that for the first 4 letters of the alphabet there are  $26 \times 25 \times 24 \times 23 = 358,800$  possibilities.
- After we have placed “A” – “Y” there is only one location left for “Z”.
- So we have that the total number of possibilities is

$$26 \times 25 \times 24 \times 23 \times \cdots \times 2 \times 1 = 26!$$

- How big is 26!?

1!	1
2!	2
3!	6
4!	24
5!	120
6!	720
8!	40,320
10!	3,628,800
20!	2,432,902,008,176,640,000
26!	403,291,461,126,605,635,584,000,000 $\approx 10^{26}$

Get the idea?

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## SOCRATIVE QUIZ - PartIII\_Practice\_Quiz2

IUZGAZ34E

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The Russian alphabet consists of 33 letters.

1. If a substitution cipher is used for the Russian alphabet instead of our alphabet, will there be
  - (a) more
  - (b) less
  - (c) the same numberpossible solutions to the encrypted message?
2. Which of the following indicates the number of possible keys for a substitution cipher using the Russian alphabet?
  - (a)  $26 \times 25 \times 24 \times \cdots \times 2 \times 1$
  - (b)  $33 \times 26 \times 25 \times 24 \times \cdots \times 2 \times 1$



(c)  $33 \times 32 \times 31 \times \cdots \times 2 \times 1$

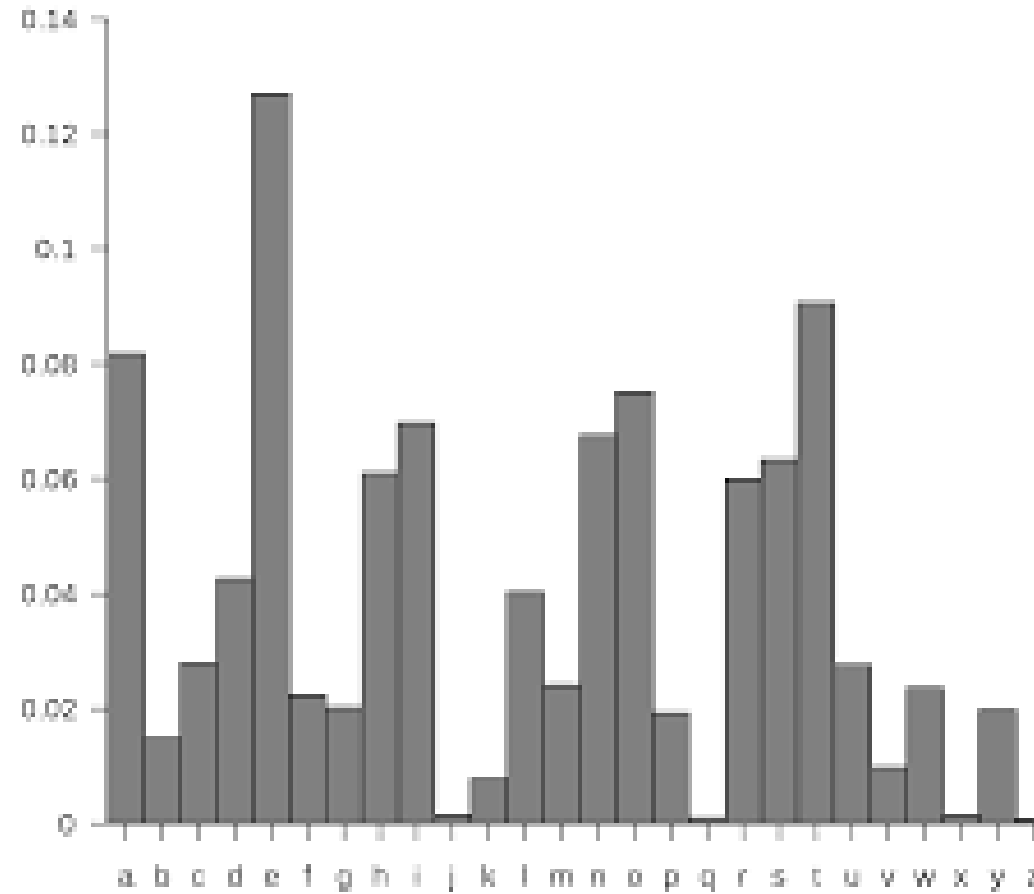
(d)  $(26 \times 25 \times 24 \times \cdots \times 2 \times 1)/33$

- Even with this simple substitution cipher the possibilities are too large to do an exhaustive search with a computer algorithm in a reasonable amount of time.
- What can we do instead?
- In any language certain letters (and words) occur more commonly than others.
- Before we said that in English the most commonly occurring letter is “E” and the second most commonly occurring letter is “T”, the third most is “A”. We can expand on this. Of course this type of frequency analysis works best for longer messages.
- If you were encrypting a code you probably wouldn't help someone trying to decrypt it by showing where one word ended and the next one started. If we assume there are no spaces indicated between words then one has to encode a letter or symbol for a space. This letter/symbol should occur most frequently.

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## Frequency plot for letters in the English language

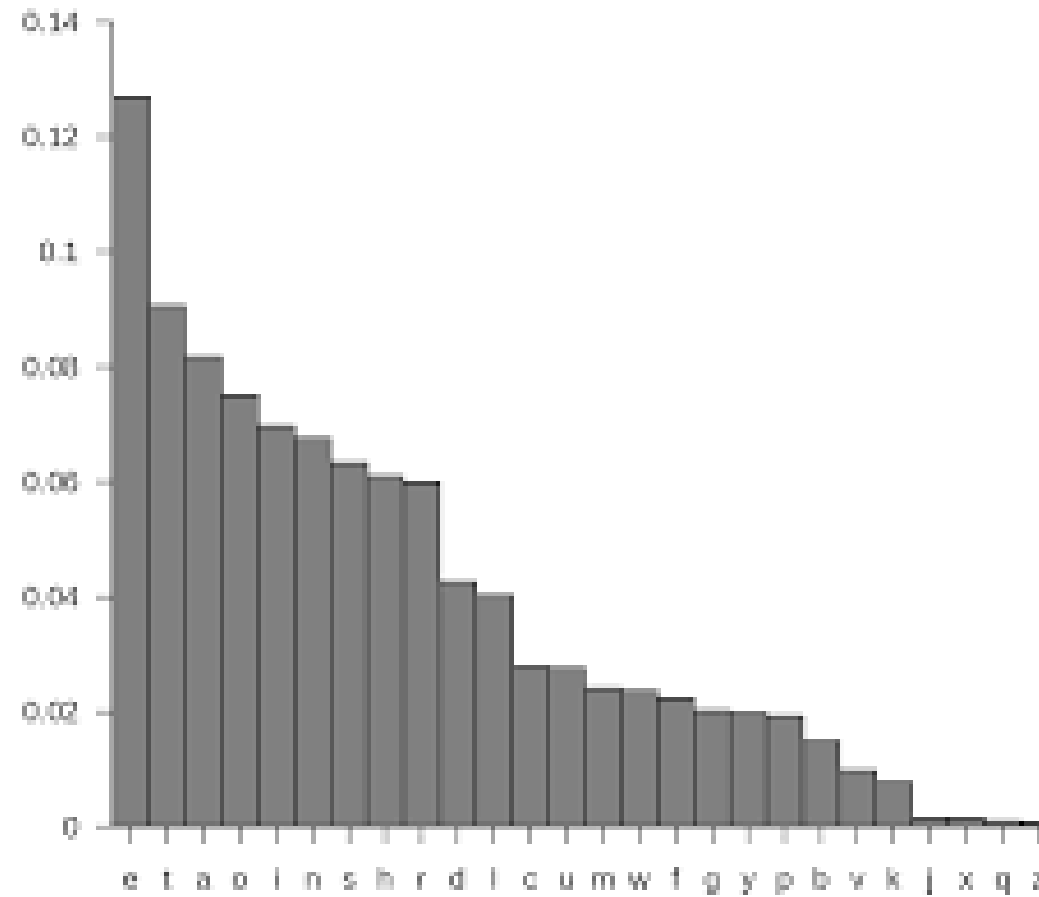
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## Frequency plot in order of highest occurring to least occurring

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**Example** Assume that you are on **Wheel of Fortune** and the following vowels and consonants have been guessed.

Vowels: E, A

Consonants: N, S, T, R

1. Use the letter frequency plot to make an “educated” guess for the next vowel.
2. Use the letter frequency plot to make an “educated” guess for the next consonant.

**Example - explanation** Assume that you are on **Wheel of Fortune** and the following vowels and consonants have been guessed.

**Vowels: E, A**

**Consonants: N, S, T, R**

1. Use the letter frequency plot to make an “educated” guess for the next vowel.

From the plot, the order of the vowels from most frequent to least frequent is

E, A, O, I, U, Y

so if “E” and “A” have already been guessed, the next most frequent vowel is “O”.

2. Use the letter frequency plot to make an “educated” guess for the next consonant.

From the plot, the order of the consonants starting with most frequent is

T, N, S, H, R, D, L, C, ...

so if “N”, “S”, “T” and “R” have already been guessed, the next most frequent vowel is “H”.

## How can we use this frequency information to assist in decrypting a code?

- We first make a frequency count for the letters in the encrypted code. All we have to do is count how many times the letter appears in the encrypted message and tabulate or plot this.
- For simplicity we will only consider ciphers that use letters, i.e., no other symbols will be used. In addition, we assume that a space is encrypted with a letter of the alphabet. Of course this means that one letter of the alphabet doesn't appear in the plaintext.
- After we have the frequency of the encrypted letters we “guess” that the most frequent occurring encrypted letter is the space. The next most frequent encrypted letter might be “E”.
- Of course the frequency plot for the encrypted letters might not directly correspond to those in the English language because these were determined by analyzing lots of texts, not just one passage. Later we will also use frequency of English words and letter combinations to decrypt the message.
- Of course a frequency count is more useful for longer messages than very short messages.

**Example** Make a letter frequency table for the following encrypted message. The message is given in groups of 5 letters but this does not indicate spaces in the message. The space is encrypted with a letter of the alphabet.

BMCA X TPAMH BMAPN BCANO  
 AHLLA BMHBA BMCDA TPBCP  
 RAUDA TPBCV FCBTN PAIMT  
 FMABM CDARV CHKAP NBANO

We now count how many times the encrypted letter “A” appears, how many times “B” appears, etc.

A	15	E	0	I	1	M	8	Q	0	U	1	Y	0
B	11	F	2	J	0	N	5	R	2	V	2	Z	0
C	8	G	0	K	1	O	2	S	0	W	0		
D	3	H	4	L	2	P	7	T	5	X	1		

The letter “A” occurs the most often and so as a first guess we assume that it encrypts

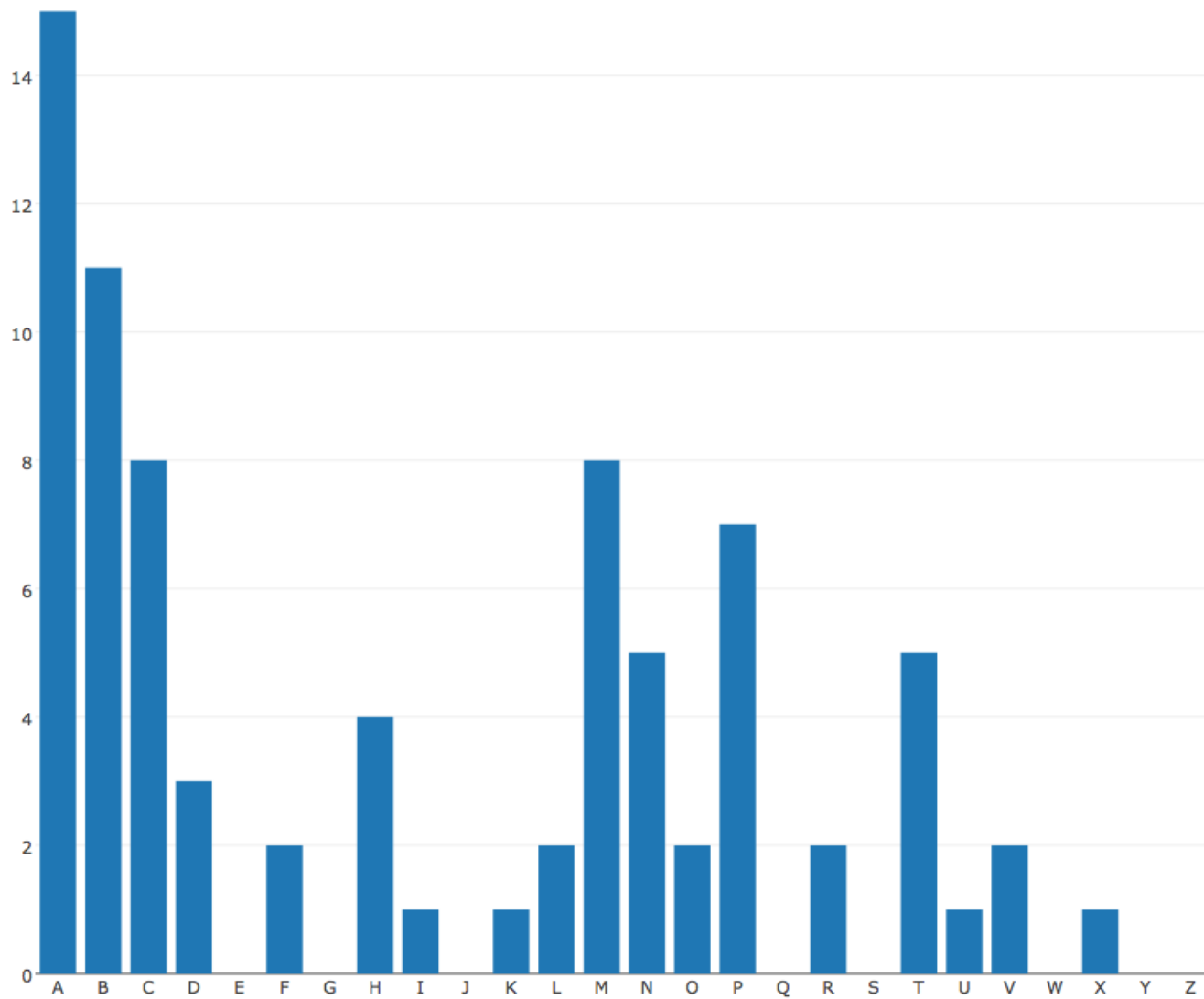


a space so the word breaks are as follows.

BMC	XTP	MHBM	PNBC	NO
HLL	BMHB	BMCD	TPBCPR	
UD	TPBCVFCBTNP	IMTFM		
BMCD	RVCHK	PNB	NO	

**Example** Using the frequency table from the previous example, use PLOTLY to make a frequency plot.

A	15	E	0	I	1	M	8	Q	0	U	1	Y	0
B	11	F	2	J	0	N	5	R	2	V	2	Z	0
C	8	G	0	K	1	O	2	S	0	W	0		
D	3	H	4	L	2	P	7	T	5	X	1		



**Example.** Use the frequency table below for an encrypted message to answer the following questions.

A 1	E 2	I 5	M 3	Q 2	U 3	Y 7
B 7	F 2	J 9	N 1	R 1	V 5	Z 4
C 4	G 3	K 12	O 5	S 11	W 1	
D 6	H 1	L 15	P 9	T 3	X 20	

1. Which letter do you believe encrypts a space?
2. What letters would you guess first that encrypts the letter "E"?

- So far we know how to make a frequency table or plot for the letters occurring in the encrypted message.
- Another strategy that is useful in decryption is to look at the most commonly occurring words in the English language.
- When you do a Google search for “most common English words” you may find that different websites may have slightly different orderings of words. This is because different texts (and number of texts) have been used to count the number of occurrences of words.
- For the information in the next two slides we have used the website [www.world-english.org](http://www.world-english.org)
- On the next slide we list the English words starting with the most frequent.
- For our application it is more useful to list the word frequency per word length starting with the most frequent. In the following slide we do this. For example, for a 1-letter word there are only two choices – “a” and “l”. The word “a” occurs more often than the word “l”. For other words lengths there are more possibilities than we listed in the table but the most common ones are noted here.

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## Top 100 Most Commonly Used Words in English Language

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Rank	Word	Rank	Word	Rank	Word	Rank	Word	Rank	Word
1	the	21	be	41	your	61	them	81	did
2	of	22	at	42	when	62	would	82	my
3	to	23	one	43	up	63	write	83	sound
4	and	24	have	44	use	64	like	84	no
5	a	25	this	45	word	65	so	85	most
6	in	26	from	46	how	66	these	86	number
7	is	27	or	47	said	67	her	87	who
8	it	28	had	48	an	68	long	88	over
9	you	29	by	49	each	69	make	89	know
10	that	30	not	50	she	70	thing	90	water
11	he	31	but	51	which	71	see	91	than
12	was	32	some	52	do	72	him	92	call
13	for	33	what	53	their	73	two	93	first
14	on	34	there	54	time	74	has	94	people
15	are	35	we	55	if	75	look	95	may
16	with	36	can	56	will	76	more	96	down
17	as	37	out	57	way	77	day	97	side
18	I	38	other	58	about	78	could	98	been
19	his	39	were	59	many	79	go	99	now
20	they	40	all	60	then	80	come	100	find



**Example.** For a previous example we did a frequency analysis on an encrypted message and determined where the words begin and end by deciding which letter (“A”) encrypted a space. We have the following encrypted message with the corresponding frequency table. With some given hints we want to see how to partially decrypt this message.

BMC                    XTP   MHBM        PNBC   NO  
 HLL                    BMHB   BMCD   TPBCPR  
 UD   TPBCVFCBTNP   IMTFM  
 BMCD                    RVCHK        PNB                NO

A	15	E	0	I	1	M	8	Q	0	U	1	Y	0
B	11	F	2	J	0	N	5	R	2	V	2	Z	0
C	8	G	0	K	1	O	2	S	0	W	0		
D	3	H	4	L	2	P	7	T	5	X	1		

Now the most frequently occurring letters are “B” , “C” and “M”. If we look at the first encrypted word we have **BMC** and two other places we have **BMCD** so a good



starting guess is

$B \rightarrow T$  and  $C \rightarrow E$

What would you guess that “M” encrypts and “H” encrypts?

We have the 3-letter encrypted word **BMC** and so far we have part of it decrypted to give **TME** so we make the guess “M” encrypts “H”.

We have a 4-letter word **BMHB** and so far we have **THHT**. Clearly **H** has to encrypt a vowel. We know that **C** encrypts “E” so it is either “A”, “I”, “O” or “U”. The only one that makes sense is **THAT** so “H” encrypts “A”.

What would you guess that “L” encrypts?

We have the 3-letter encrypted word **HLL** and we know that “H” is “A”. If we look at the common 3-letter words beginning with “A” and having a double letter next we see that “ALL” is the only choice so “L” must encrypt itself.

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## Other Simple Substitution Ciphers

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So far we have only mixed up the alphabet to obtain the key but we could also use a mixture of letters, numbers and symbols. For example, the key could be

A B C D E F G H I J K L M N O P Q R S T U V W X Y Z

B 1 & A 8 4 H \* T ! 5 X N + L = 7 V W U 3 G 6 S Q R

To encrypt a message we proceed as usual. For example, to encrypt **GO NOLES** we see that a “G” is represented by an “H”, an “O” by an “L”, an “N” by a “+”, etc. Now if we want to encrypt the “space” then we choose a letter or symbol that hasn’t been used in the key. For example the number “2” hasn’t been used so if we use that the encrypted message is

**HL2+LX8W**

This is still a monoalphabetic substitution cipher and we can still use frequency analysis

of English to decode the message.

Now let's see how to decrypt a message. Consider the encrypted message given in 5 symbol blocks.

\*BG8C +LC48 BVCL4 C=8V4 8&UTL  
 +CQL3 C6TXX C+8G8 VCV8B &\*CTU

We create the frequency table for the encrypted message

B 3	C 9	G 2	L 4	Q 1	U 2	V 4
X 2	3 1	4 3	* 2	+ 3	= 1	& 2
8 7						

We see that "C" occurs the most often and so it probably encrypts a space. We have the message

\*BG8 +L 48BV L4 =8V48&UTL+  
 QL3 6TXX +8G8V V8B&\* TU

The number "8" is by far the next most frequent letter so we substitute an "E" and

see if it seems to make sense.

\*BGE +L 4EBV L4 =EV4E&UTL+  
QL3 6TXX +EGEV VEB&\* TU

If we concentrate on the 2-letter words we have three

+L L4 TU

The most common 2-letter words are

of, to, in, is, it, he, on, as, be , at , or, by, we, up, an, do, if, so, go , no

Notice that L is the second letter of +L and the first letter of L4 and of the listed 2-letter words only an "O" works. We have

\*BGE +O 4EBV O4 =EV4E&UTL+  
QO3 6TXX +EGEV VEB&\* TU

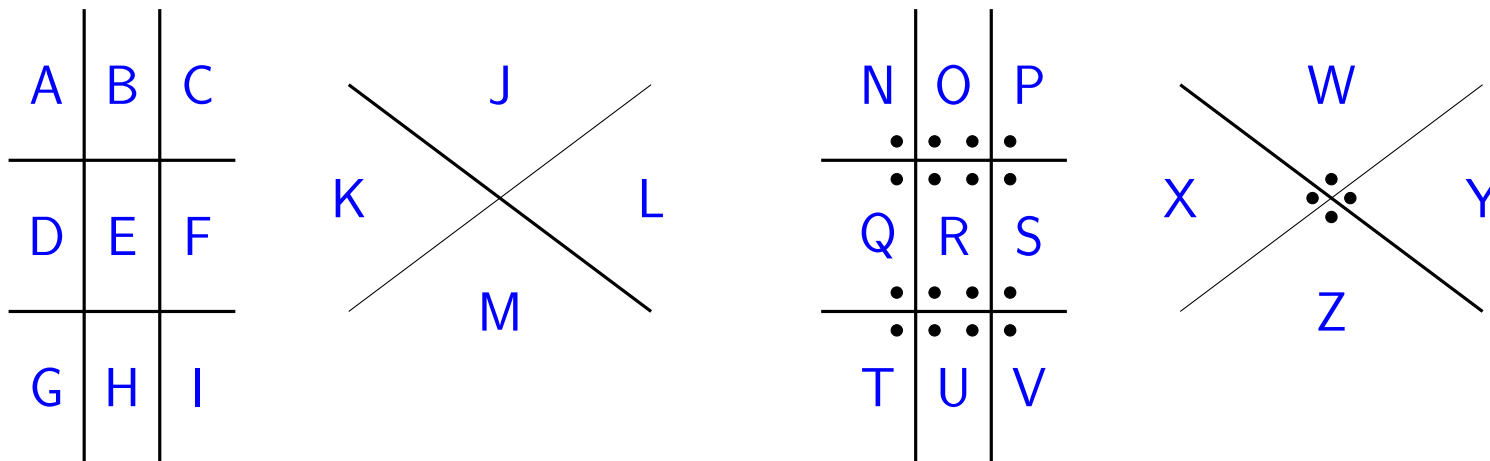
The most frequently occurring letter in English is "E" which we have decoded and the next is "T" and then "A". The letter V occurs 4 times, the letter B 3 times, the number 4 3 times, + three times. So these are all candidates for "T" and "A".

You can practice decoding by finishing this exercise. If you get stuck the key is just the one that is given above.

## The Pigpen Cipher

Another simple substitution cipher is the Pigpen cipher also known as the Masonic cipher, Napoleon cipher, tic-tac-toe cipher. It is interesting because it is a geometric substitution cipher and doesn't just use letters or numbers it uses uncommon symbols.

To determine the key we write the letters of the alphabet on 4 grids. For example,



How do we encrypt letters using this key?

An A is encrypted as 

A **B** is encrypted as 

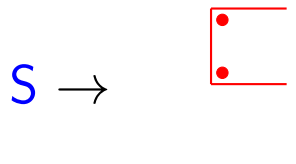
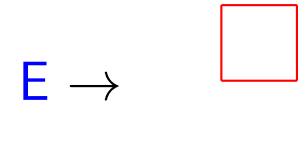
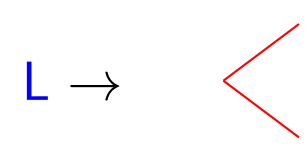
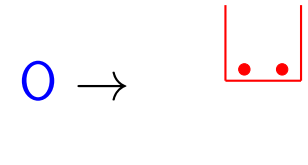
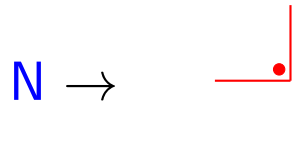
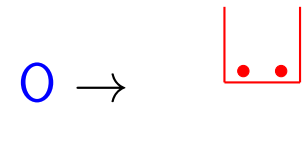
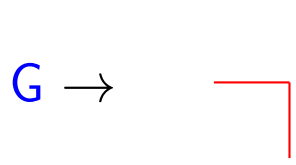
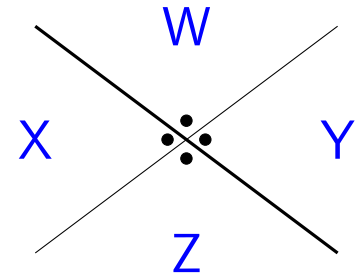
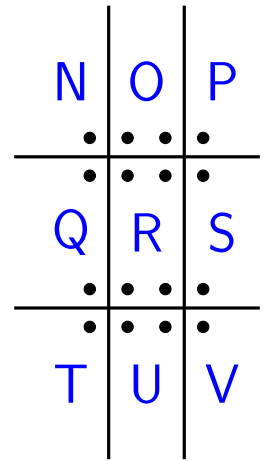
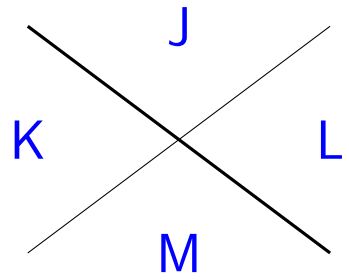
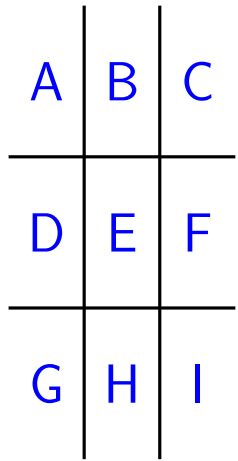
An **M** is encrypted as 

A **P** is encrypted as 

A **Z** is encrypted as 

**Example.** Use the key above to encrypt **GO NOLES**.

# Explanation





To decrypt a message using the Pigpen cipher we do exactly as before. We make a frequency table/plot of the symbols contained in the encrypted message and then use frequencies of letters and words in the English language.

Even if we mixed up the order of the alphabet in the key above, we still approach the decryption in the same way and it still takes the same amount of work to decipher.

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## Weaknesses of Monoalphabetic Ciphers

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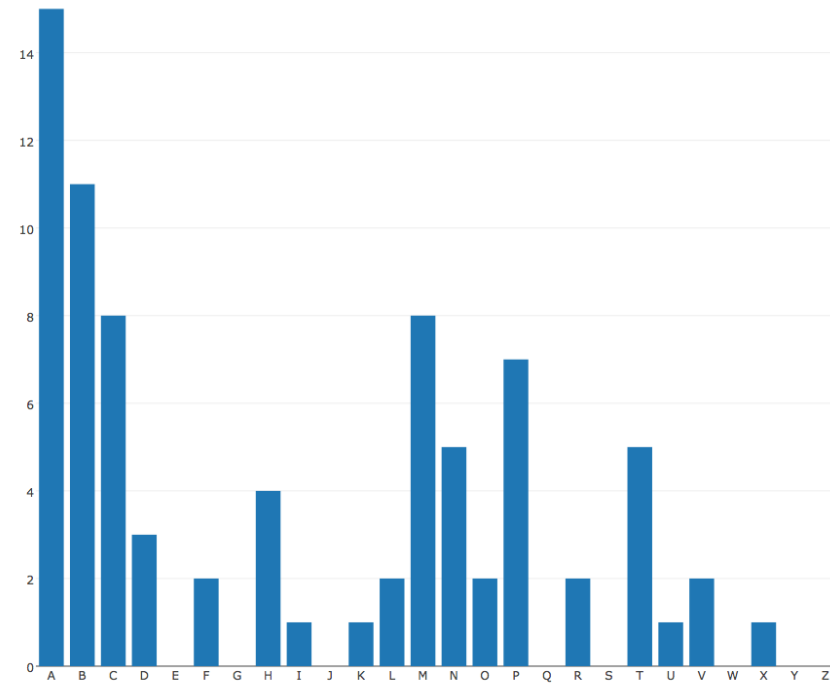
- In the simple substitution ciphers we take the alphabet or set of symbols to be **fixed** throughout the encryption process. For example, “A” might always be encrypted as “P”, “B” always encrypted as “3”, etc. These are **monoalphabetic** ciphers.
- All monoalphabetic ciphers are **very susceptible** to decryption.
- Why are they relatively easy to decrypt? Because although the symbols in the key themselves change, their **frequency** does not. So frequency analysis of the language of the original plaintext can be used. This frequency analysis approach was first documented by an Arabic mathematician in the 9th century.

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## SOCRATIVE QUIZ - PartIII\_Quiz2

IUZGAZ34E

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1. The frequency plot above represents the frequency of occurrence of letters in an encrypted message. Assuming the “space” is encrypted, what letter do you think represents it?
  - a. A
  - b. B
  - c. C
  - d. M
  
2. The frequency plot above represents the frequency of occurrence of letters in an encrypted message. How many letters appear 3 times?
  - a. 1
  - b. 2
  - c. 3
  - d. 4
  
3. The frequency plot above represents the frequency of occurrence of letters in an encrypted message. How many letters appear once?
  - a. 1

b. 2

c. 3

d. 4

4. The frequency plot above represents the frequency of occurrence of letters in an encrypted message. How many letters of the alphabet don't appear in the encrypted message?

a. 5

b. 6

c. 8

d. 9

5. Which of the following ciphers is NOT a monoalphabetic cipher?

a. Caesar Cipher with shift 2

b. Atbash Cipher

c. Pigpen Cipher

d. None of the above

6. If we use a simple substitution cipher using 26 letters in random order, how many possibilities are there for the key?

- a. 25
- b. 25!
- c. 26
- d. 26!

7. If we do a letter frequency table or plot for an encrypted message where a space is NOT encrypted, what do typically the two most frequent letters represent?

- a. A, E
- b. E, N
- c. A, T
- d. E, T

8. The most common word in the English language is \_\_\_\_\_

- a. A
- b. I
- c. AN
- d. THE

9. If an encryption key contains both letters and numbers in a random fashion then it is more difficult to decrypt than one which just uses letters in random order.

10. A weakness of all monoalphabetic ciphers is that one can use frequency analysis for the language the plaintext is written in to decode the message.

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## Goals for lecture

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1. In this lecture we will do an example of completely decrypting a message which was encrypted by using a monoalphabetic substitution cipher. We will use a strategy which uses frequency analysis and describe the approach in a step-by-step manner.
2. We will look at ways to improve the complexity of simple substitution ciphers.



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## Decryption of a Message Using a Monoalphabetic Substitution Cipher

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Assume this encrypted message has 53 5-letter groups with no punctuation and assume that we know that a letter represents a space.

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**Step 1.** Determine which letter encrypts a space by doing a frequency analysis and then write the message in terms of actual word length.

MJZYB LGESE CNCMQ YGXY S PYZD Z PMYGI IRL LC  
PAYCK YKGWZ MCWZK YFRCM ZYVCX XZLZP MYXLG  
WYTJS MYGPZ YWCAJ M YCWS ACPZY XGLYZ HSWBN  
ZYXZT YTGRN VYMJC POYMJ SMYCX YMJZL ZYSLZ  
YMTZP MQYMJ LZZYB ZGBNZ YCPYS YLGGW YMJZP  
YMJZL ZYCKY SPYZD ZPKYI JSPIZ YMJSM YMJZL  
ZYSLZ YMTGY GXYMJ ZWYTC MJYMJ ZYKSW ZYECL  
MJVSQ YERMY MJCKY CKYKG

We now perform the tedious task of counting how many times “A” appears, how many times “B” appears, etc. to get the following table.

A	3	E	4	I	4	M	27	Q	3	U	0	Y	49
B	4	F	1	J	17	N	4	R	4	V	3	Z	33
C	18	G	14	K	9	O	1	S	14	W	9		
D	2	H	1	L	14	P	13	T	6	X	8		

- Since Y occurs most often by far we assume that it represents a space.
- We now replace Y with a space to see the word length.

MJZ	BLGESECNCMQ	GX	SP	ZDZPM	GIIRLLCPA
CK	KGWZMCWZK	FRCMZ	VCXXZLZPM	XLGW	TJSM
GPZ	WCAJM	CWSACPZ	XGL	ZHSWBNZ	XZT
TGRNV	MJCPO	MJSM	CX	MJZLZ	SLZ
MTZPMQ	MJLZZ	BZGBNZ	CP	S	LGGW
MJZP	MJZLZ	CK	SP	ZDZPK	IJSPIZ
MJSM	MJZLZ	SLZ	MTG	GX	MJZW
TCMJ	MJZ	KSWZ	ECLMJVSQ	ERM	MJCK
CK	KG				

---

**Step 2.** We now list the encrypted words by length (for length 1-4) and their number of occurrences which is given in parentheses in the table below.

1-letter words	S (1)					
2-letter words	GX (2)	SP(2)	CK(3)	CX(1)	CP(1)	KG(1)
3-letter words	MJZ(2) ERM(1)	GPZ(1)	XGL(1)	XZT(1)	SLZ(1)	MTG (1)
4-letter words	XLGW(1) TCMJ (1)	TJSM(1) KSWZ(1)	MJSM(2) MJCK(1)	LGGW(1)	MJZP(1)	MJZW(1)

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**Step 3.** We now look at the next most frequently appearing letters and use frequency analysis of letters and words to begin the decoding process. Start with short words.

- The next two most frequently appearing letters are **Z** (33 appearances) and **M** (27 occurrences) so these are candidates for **E** and **T**. We could try each of these and see if we could guess some words. In fact, we know that “the” is the most common 3-letter word and both “T” and “E” appear in it so it’s a safe guess that a 3-letter word with **M** and **Z** is “THE”; of course it could be other words too (like “TIE”). The encrypted word **MJZ** appears twice and all other 3-letter words occur once so a good beginning guess is that

**M** → **T**

**J** → **J**

**Z** → **E**

- We also note that there is a single 1-letter word which should be either “A” or “I” but we can’t make a choice from this. However, notice that we have a four-letter word **THST** so we suspect it is **THAT**. Thus **S** → **A**

Below is the partially decrypted message. As before, a red letter means encrypted and a blue letter is decrypted.

THE	BLG	AECNCTQ	GX	AP	EDEPT	GIIRLLCPA
CK	KGW	ETCWEK	FRCTE	VCXXE	LEPT	XLGW THAT
GPE	WCAHT	CWAACPE	XGL	EHSWBNE	XET	
TGRNV	THCPO	THAT	CX	THELE	ALE	
TTZPMQ	THLEE	BEGBNE	CP	A	LGGW	
THEP	THELE	CK	AP	EDEPK	IHAPIE	
THAT	THELE	ALE	TTG	GX	THEW	
TCTJ	THE	KAW	E	CLTHVSQ	ERT	THCK
CK	KG					

- We have the three-letter word **ALE**. We check the frequency of words by length and see that this is probably **ARE** making **L** an **R**.
- Also we have a word **AP**. Checking the frequency of two-letters words we have

“at”, “an” and “as” but it can’t be “at” because we know that **M** is **T**. Thus **P** must be **S** or an **N**. However, it can’t be an **S** because look at the word **THE**P****. If it is an **S** then this becomes **TH**S**** whereas if it is an **N** then it becomes **TH**E**N******. Thus **P** represents an **N**.

Making the replacements **S** → **A**, **L** → **R**, and **P** → **N**. Our message becomes (where the blue letters are plaintext and the red are encrypted)

THE BRGEAECNCTQ GX AN EDENT GIIRRCNA  
CK KGWETCWEK FRCTE VCXXERENT XRGW THAT  
GNE WCAHT CWAACNE XGR EHAWBNE XET  
TGRNV THCNO THAT CX THERE ARE  
TTZNMQ THREE BEGBNE CN A RGGW  
THEN THERE CK AN EDENK IHANIE  
THAT THERE ARE TTG GX THEW  
TCTJ THE KAWE ECRTHVAQ ERT THCK  
CK KG



So far we have decrypted the following letters

Encrypted	J	L	M	P	S	Z
	↓	↓	↓	↓	↓	↓
Decrypted	H	R	T	N	A	E

- Now we look at the remaining 2-letter words which we haven't decrypted; their number of occurrences is in parentheses. .

CK(3), CX(1), GX(2), KG(1)

- The most common 2-letter words (in order) are

of to in is it he on as be at or by we up an do  
if so go no

- We already know that T is represented by M so we can rule out "to", "it", "at" so that leaves us with

of is in he on as be or by we up an do if so go no

- We also know that **P** decrypted is **N** and **Z** decrypted is **E** so we can rule out any of these 2-letters words which contain “N” or “E”, i.e., “in”, “on”, “an”, “no” and “he”, “be”, “we” which leaves us with the top ten possible 2-letter words

of    is    as    or    by    up    do    if    so    go

- Two of our code words start with a “C” (including the one that occurs the most) and 2 of our top English words begin with an “O” (including the most common and 4th most frequent) so let’s guess that “C” is an “O” which gives

**CK** , **CX** , **GX** , **KG**  $\implies$  **OK** , **OX** , **GX** , **KG**

This suggests that **K**, **X** represent **F** , **R** but we also have a 2-letter word **KG** so it doesn’t look like **K** is either “F” or “R” because there are no common 2-letter word beginning with “F” or “R”. So it looks like our guess that “C” represents an “O” is incorrect!

- So probably **GX** or **KG** represents **OF**. Now **KG** is the last word of the message so it doesn’t make sense that **KG** would be **OF** so we guess that **G** represents **O** and **X** represents **F**. To reinforce this assumption look at the word **VCXXERENT**; if **X** represents **F** then this is probably **DIFFERENT**.
- The last encrypted word of the message is **KG** and since we guessed that **G** rep-

resents O we have KO. What 2-letter words end with “O” that could be the last word of the message? Continuing down the list of common 2-letter words we find the following words that end in “O” and don’t contain the letters T, H, E, A, R, N, O, F which we have already decoded.

do, so, go

It’s not clear how to proceed from here so let’s move on to the other 2-letter words.

- The other 2-letter words are CK and CF. Remember that CK occurs the most frequently so our next most frequent 2-letter word is “IS” which would mean that C → I and K → S. This would mean that CF is IF and KO is SO; both make sense.

So far we have decrypted the following.

Encrypted	C	G	J	K	L	M	P	S	X	Z
	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓
Decrypted	I	O	H	S	R	T	N	A	F	E

THE BROEAENITQ OF AN EDENT OCCRRRINA  
IS SOWETIWES FRITE VIFFERENT FROW THAT  
ONE WIAHT IWAAINE FOR EHAWBNE FET  
TORNV THINO THAT IF THERE ARE  
TTENTQ THREE BEOBNE IN A ROOW  
THEN THERE IS AN EDENS IHANIE  
THAT THERE ARE TTG OF THEW  
TITJ THE SAWE EIRTHVAQ ERT THIS  
IS SO

## SOCRATIVE QUIZ - PartIII\_Quiz7

IUZGAZ34E

1. Looking at the partially decrypted words **SOWETIWES** and **SAWE**, what letter do you think that “W” encrypts?
2. Looking at the partially decrypted word **VIFFERENT** , what letter do you think that “V” encrypts?
3. Looking at the partially decrypted word **EDENT** , what letter do you think that “D” encrypts?

Using the facts (from the quiz) that

$W \rightarrow M$        $V \rightarrow D$        $D \rightarrow V$

we have so far decrypted the following letters which gives the partially decrypted message below.

Encrypted	C	D	G	J	K	L	M	P	S	V	W	X	Z
	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓
Decrypted	I	V	O	H	S	R	T	N	A	D	M	F	E

THE BROEAENITQ OF AN EVENT OIIRRRINA  
IS SOMETIMES FRITE DIFFERENT FROM THAT  
ONE MIAHT IMAAINE FOR EHAMBNE FET  
TORNV THINO THAT IF THERE ARE  
TTENTQ THREE BEOBNE IN A ROOM  
THEN THERE IF AN EDENS IHANIE  
THAT THERE ARE TTG OF THEM  
TITJ THE SAME EIRTHDAQ ERT THIS  
IS SO

From the word **MIAHT** we see that **A** → **G**

From the words **THAT** and **FET** we see that **T** → **W**

This means that the word **TTENTQ** becomes **TWENTQ** so **Q** → **Y**

Now the word **EIRTHDAQ** becomes **EIRTHDAY** so **E** → **B**.

This makes sense because then **ERT** becomes **BRT** so **R** → **U**.

Encrypted

**A C D E G J K L M P Q R S T V W X Z**

Decrypted

**G I V B O H S R T N Y U A W D M F E**



THE PROBABILITY OF AN EVENT OCCURRING  
IS SOMETIMES QUITE DIFFERENT FROM WHAT  
ONE MIGHT IMAGINE FOR EXAMPLE FEW  
WOULD THINK THAT IF THERE ARE  
TWENTY THREE PEOPLE IN A ROOM  
THEN THERE IS AN EVEN CHANCE  
THAT THERE ARE TWO OF THEM  
WITH THE SAME BIRTHDAY BUT THIS  
IS SO

We need to know the following 6 encrypted letters to finish decrypting the message.

F, H, B, N, O, I

From the word FUIITE we see that F → Q.

From the words OIIURRING and IHANIE we see that I → C.

From the word WOUND we see that N → L.

Now we have the word BEOBLE so B → P.

Now the word EHAMBNE now becomes EHAMPLE so H → X.

Lastly we have the word THINO we see that O → K.

The final decrypted message is :

THE PROBABILITY OF AN EVENT OCCURRING  
IS SOMETIMES QUITE DIFFERENT FROM WHAT  
ONE MIGHT IMAGINE FOR EXAMPLE FEW  
WOULD THINK THAT IF THERE ARE  
TWENTY THREE PEOPLE IN A ROOM  
THEN THERE IS AN EVENS CHANCE  
THAT THERE ARE TWO OF THEM  
WITH THE SAME BIRTHDAY BUT THIS IS  
SO

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## Making the Cipher more Complex

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- In the previous simple substitution ciphers the cipher alphabet is **fixed** throughout the encryption process. For example, “A” might always be encrypted as “P”, “B” always encrypted as “3”, etc. These are called **monoalphabetic** ciphers.
- All monoalphabetic ciphers are **very susceptible** to decryption.
- What if we allowed “A” to be encrypted as “P” in one place but as “T” in another place. These types of methods are called **polyalphabetic** ciphers. An example is the Alberti cipher which was developed in the 15th century.
- Polyalphabetic ciphers make decryption much more difficult.
- What would a key look like for such a cipher? First, we look at the Alberti cipher.

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## Alberti Cipher

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Recall that the Alberti cipher used a physical apparatus which consisted of two disks, one fixed and one movable.



For simplicity, assume that the large outer disk consist of the letters

ABCDEFGHIJKLMNOPQRSTUVWXYZ

which will represent the plaintext. Assume the inner disk consists of

abcdefghijklmnopqrstuvwxyz

which will be the encrypted letters.

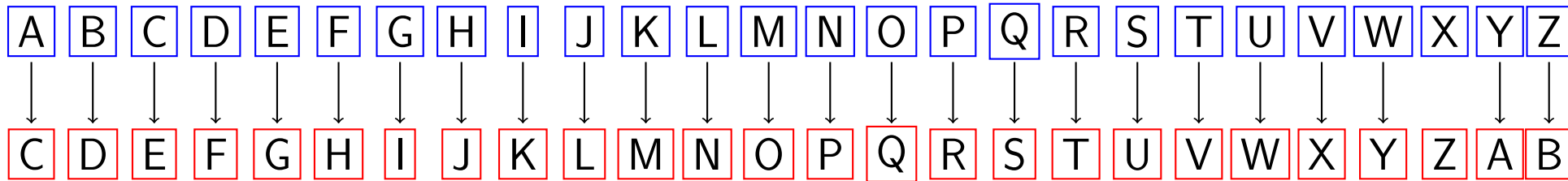
Assume you turn the disk so that “A” lines up with “a”, “B” lines up with “b” etc. If you rotate the disk by 2 letters then “A” lines up with “c”, “B” lines up with “d”, i.e., you have done a Caesar shift to the right with shift two. Now choose a number, say 4, which we will call the period. You encrypt the first four letters of the plaintext and then you rotate the disk by one letter which gives a shift of three from the original. You encrypt the next four letters of the plaintext with this Shift 3 encryption and then rotate the disk again.

**Example** Encrypt the message

GO SEMINOLES

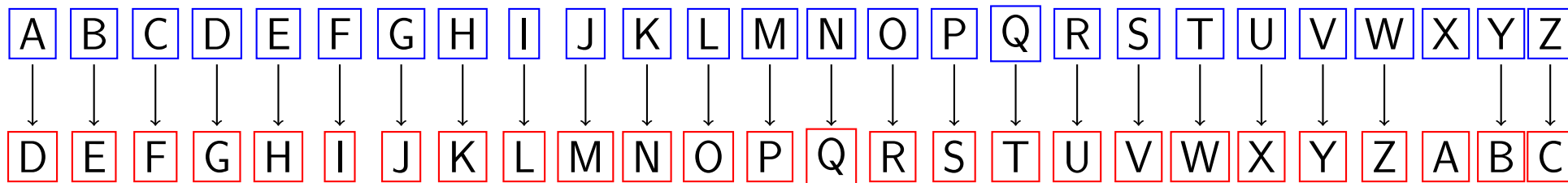
using the Alberti cipher with an original shift of 2, a period of 3 (i.e., every 3 letters rotate disk by 1) assuming the disks are set up as described above.

We begin by using a Caesar shift 2 to encrypt the first three letters of the message  
GO S



We have IQ UEMINOLES.

To encrypt the next 3 letters EMI we now rotate the disk by 1 to use a Caesar shift 3.



to encrypt EMI as HPL so that we have IQ UHPLNOLES.

Now we rotate the disk by 1 and use a Caesar 4 shift to encrypt **NOL** as **RSP**. Note that the “O” in **GO** and in **SEMINOLES** are encrypted differently.

Now we rotate the disk by 1 and use a Caesar 5 shift to encrypt **ES** as **JX** to get the encrypted message

**IQ UHPLRSPJX**

It is important to realize that if we do a frequency table for the letters occurring in the encrypted message, it won't help us because “O” is encrypted as “Q” in one place and as “S” in another. Likewise “S” is encrypted as “U” in the first occurrence and as “X” in the last occurrence. This is what makes polyalphabetic ciphers more difficult to break.



---

## Vigenere Cipher

---

- In this cipher we use a text string such as a word as a key.
- Let's say the key is the word **NOLES**. We look at the numerical position in the alphabet of each letter in the word.

Letter	Position	Letter	Position	Letter	Position	Letter	Position	Letter	Position
A	1	B	2	C	3	D	4	E	5
F	6	G	7	H	8	I	9	J	10
K	11	L	12	M	13	N	14	O	15
P	16	Q	17	R	18	S	19	T	20
U	21	V	22	W	23	X	24	Y	25
Z	26								

We have

**N** → 14,   **O** → 15   **L** → 12   **E** → 5   **S** → 19

Our key is

14 15 12 5 19

- Now assume that the message we want to encrypt is

ATTACK FROM SOUTHWEST

- To encrypt the message we write below it the key 14,15,12,5,19 cyclically.



- We shift “A” by 14 letters so it is replaced by the 15th letter in the alphabet which is “O”
- We shift the first “T” by 15 characters. Note that “T” is the 20th letter in the alphabet so if we shift it by 6 characters we get “Z” so we need to shift it by 9 more characters and the 9th letter of the alphabet is “I”
- The next “T” is shifted by 12 characters so we need the 6th letter of the alphabet so it is encrypted as “F”.
- The next “A” is shifted by 5 characters and it is encrypted as “F”
- Continuing in this manner the encrypted message is then

OIFFVYUDTFGDGYAKTEY

- Again, frequency analysis doesn’t help us to decode the message because, e.g., in one place “A” is replaced by “O” and in another place by “F”.
- This code was easy to use because all you have to do is give the key word to the recipient and you can change the key word often. It was very hard to break before computers.



## What if we are given the key and want to decrypt a message?

- Suppose we have the key word **PARTY** in our possession.
- We want to decrypt the message

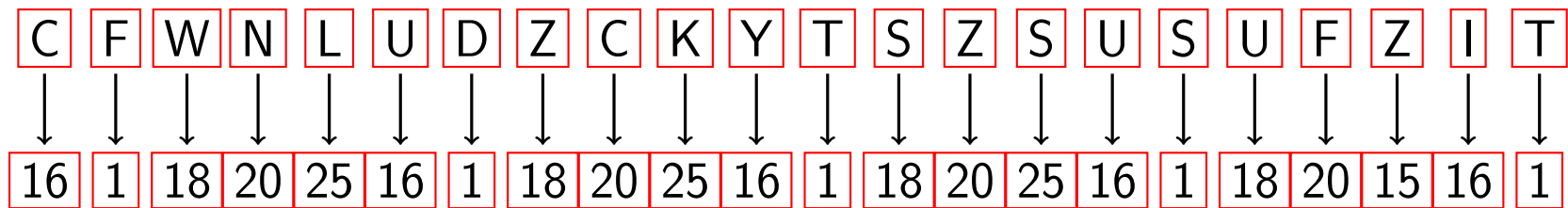
**CFWNLUDZCKYTSZSUSUFZIT**

- We first associate each letter in the key word with its numerical position in the alphabet.

$P \rightarrow 16, \quad A \rightarrow 1 \quad R \rightarrow 18 \quad T \rightarrow 20, \quad Y \rightarrow 25$

Our key is now **16 1 18 20 25**

- Now under each letter in the encrypted message write the corresponding number. To do this just write **16 1 18 20 25** in a cyclic manner.



A B C D E F G H I J K L M N O P Q R S T U V W X Y Z

- This says that the first encrypted “C” was encrypted by taking a letter of the plaintext alphabet and shifting it 16 letters to the right. So we have to shift to the **left** (or backwards) 16 letters to get the corresponding plaintext letter. **C → M**
- Now the second encrypted letter is “F” and it was encrypted by taking a plaintext letter and shifting one character to right so we shift to the left 1 character to get **F → E**
- Now “W” was formed by taking a plaintext character and shifting to the right 18 letters so we shift to the left 18 letters to get **W → E**

MEETMECHILISAFTERCLASS

MEET ME CHILIS AFTER CLASS

Note that string the **SZSUS** forms a word (**AFTER**). Clearly there are three “S” letters in the encrypted message but each one represents a different letter!





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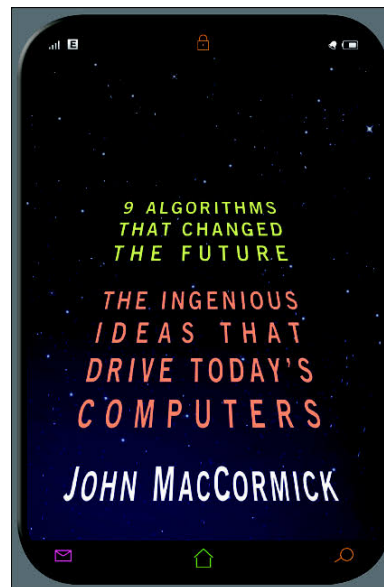
## Reading Assignment for Next Time

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Algorithm 4 in your text is

Public Key Cryptography: Sending Secrets on a Postcard

Read pages 38–43



---

## SOCRATIVE QUIZ - PartIII\_Quiz3

IUZGAZ34E

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1. Which of the following is NOT a monoalphabetic cipher?
  - a. Caesar cipher
  - b. Substitution cipher using symbols instead of letters
  - c. Atbash cipher
  - d. Vigenere cipher
2. Which of the following is a polyalphabetic cipher?
  - a. Caesar cipher
  - b. Alberti cipher
  - c. Vigenere cipher
  - d. (b) and (c)
3. The main difference between a monoalphabetic cipher and a polyalphabetic cipher is that a letter such as "A" is always encrypted the same way in a polyalphabetic

cipher but in a monoalphabetic cipher it can be encrypted by different letters in different places.

4. The reason that a polyalphabetic cipher is more difficult to decrypt than a monoalphabetic cipher is that frequency analysis can't be used.
5. In a Vigenere cipher if the key word is **FLORIDA** then to encrypt the phrase

**MEETMETONIGHTUSUALPLACE**

the first "M" is encrypted as

- a. R
- b. S
- c. F
- d. H

6. In a Vigenere cipher if the key word is **FLORIDA** then to encrypt the phrase

**MEETMETONIGHTUSUALPLACE**

the first "O" is encrypted as

- a. P

b. S

c. T

d. U

---

## Goals for this Lecture

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1. To discuss some of the differences between Classical Cryptography and Modern Cryptography
2. To introduce the concept of binary digits.
3. To introduce the two types of Keys in modern cryptography - Public and Private.
4. To introduce the two types of Encryption Systems.
5. To explain Symmetric Key Encryption and give examples.

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## Modern Cryptography

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What are the major differences between Modern Cryptography and Classical Cryptography?

- While classical cryptography manipulates traditional characters (letters, digits) directly modern cryptography operates on binary bit sequences. What does this mean?



A binary digit can only be 0 or 1. In computer talk, binary digit is often shortened to bit. The reason that a binary digit is used is because it can be represented as on/off, positive/negative, north/south, etc. Computers store their information in bits. Bits are the building blocks of computer data storage. A group of 8 bits is called a byte.

To begin to see how binary numbers correspond to our decimal numbers let's count from 1 to 20 in each system.

So 0 and 1 are easy in binary because they are the same as in our decimal system but what do we do when we want a binary number for 2? The same thing that we do when we increase 9 by 1 to get 10; so 2 in binary is just 10. Now for 3 in binary we do the same that we did when we increase 10 by 1 to get 11 so 3 is just 11 in binary. Thus 3 is the largest number we can make with 2 binary digits so for our system the largest two digit number is 99 so to increase it we go to 100. Thus 4 in binary is just 100.

Decimal	Binary	Decimal	Binary	Decimal	Binary	Decimal	Binary
1	1	6	110	11	1011	16	10000
2	10	7	111	12	1100	17	10001
3	11	8	1000	13	1101	18	10010
4	100	9	1001	14	1110	19	10011
5	101	10	1010	15	1111	20	10100

Notice how all odd numbers end in 1 in binary and even end in 0. When we say 10 in “binary talk” we say “one zero” not ten.



## Binary Examples

1. Which is larger 101001 or 101010?
2. If 20 in binary is 10100, what is 21?
3. If 20 in binary is 10100, what is 22?
4. Write the largest 2 bit binary number. Write the largest 3 bit binary number. Write the largest 4 bit binary number.
5. There are 2 numbers which can be represented as a 2-digit binary number (2 and 3); there are 4 numbers which can be represented as a 3-digit binary number; there are 8 numbers which can be represented as a 4-digit binary number. How many different numbers can be represented by a 5-digit binary number?
6. How many different numbers can a 6-digit binary number represent?

The storage on your computer is probably measured in gigabytes (GB). Now remember that a byte is 8 binary digits or bits and giga is a prefix meaning  $10^9$  so one GB is 1,000,000,000 bytes or 8,000,000,000 bits. But how much is that in our decimal system? Let's see how to convert from binary to decimal but we will use smaller numbers.

We know that our decimal number system is based on **10**; that is, we use digits from 0 to 9. For example,

$$58,913 = 3 \cdot 1 + 1 \cdot 10 + 9 \cdot 100 + 8 \cdot 1,000 + 5 \cdot 10,000$$

Mathematically we write this using exponents as

$$58,913 = 3 \cdot 10^0 + 1 \cdot 10^1 + 9 \cdot 10^2 + 8 \cdot 10^3 + 5 \cdot 10^4$$

So then each digit to the left of the decimal point increases by a power of 10.

The binary digit system is based on **2** so it uses digits from 0 to 1. So if we have a binary number it can be expressed in powers of  $2^0, 2^1, 2^2, \dots$ . For example, we have

$$100 = 0 \cdot 2^0 + 0 \cdot 2^1 + 1 \cdot 2^2 = 4$$

$$111 = 1 \cdot 2^0 + 1 \cdot 2^1 + 1 \cdot 2^2 = 7$$

$$10011 = 1 \cdot 2^0 + 1 \cdot 2^1 + 0 \cdot 2^2 + 0 \cdot 2^3 + 1 \cdot 2^4 = 1 + 2 + 0 + 0 + 16 = 19 \quad \text{in base 10}$$

Recall that we showed previously that the binary 10011 represented 19 by counting starting from 1 in binary.

So representing a binary number as a decimal number is easy.

As another example consider the 7 digit binary number 1000000

$$1000000 = 0 \cdot 2^0 + 0 \cdot 2^1 + 0 \cdot 2^2 + 0 \cdot 2^3 + 0 \cdot 2^4 + 0 \cdot 2^5 + 1 \cdot 2^6 = 2^6 = 64$$

We will need to convert binary numbers to decimal because a

---

## SOCRATIVE QUIZ - PartIII\_Practice\_Quiz?

IUZGAZ34E

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Note:

$$2^0 = 1, 2^1 = 2, 2^2 = 4, 2^3 = 8, 2^4 = 16, 2^5 = 32$$

1. Which of the following is NOT a valid binary number?

- (a) 1010111
- (b) 1010211
- (c) 1111111
- (d) 1000000

2. The next binary number after **1111** is

- (a) 1112
- (b) 10000
- (c) 10001

(d) 10010

3. What is the largest 5 bit binary number?

(a) 10000

(b) 10010

(c) 10101

(d) 11001

(e) 11111

4. What is the binary number **101** in our base 10 number system?

(a) 101

(b) 4

(c) 5

(d) 6

5. What is the binary number **1010** in our base 10 number system?

(a) 8

(b) 9

(c) 10

(d) 11

---

## Differences between Classical and Modern Cryptography

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- Classical cryptography is mainly based on “security through obscurity” while modern cryptography relies on **mathematical algorithms** for coding the information.
- In classical cryptography Keys were kept secret and only the parties encrypting and decrypting the messages know about them (in theory).
- In modern cryptography secrecy is obtained through a Key which is used to initialize an encrypting or decrypting algorithm. Even if the algorithm is known, it is impossible for an attacker to decrypt the message without a Key.
- Just like in our simple ciphers the encryption algorithm produces ciphertext given the original plaintext and the encryption Key. It’s just that now it’s a much more complicated procedure than simply shifting or substituting characters.
- Also we have a decryption algorithm that produces unique plaintext from the ciphertext and the decryption Key. For example, when a Shift Cipher was used our decryption algorithm just told us how far to shift our letters.

# Modern Cryptography Problem



The sender wants to transfer sensitive data such as a credit card number to a receiver such as Amazon in such a way that an unauthorized person (the interceptor or attacker) can't get the information. The objective is that only the sender and receiver know the sensitive data.



Here's a simple example similar to the one in your text.

Suppose you want to tell your classmate a secret right now but your younger brother is in the room and you don't want him to know the secret.

**Rule:** You are only allowed to communicate by speaking out loud; i.e., you are NOT allowed to whisper in your friend's ear.

**Simplifying Assumption:** The "secret" you want to tell your friend is the number 15.

**How can you do this?** You think of something that both you and your classmate know but your brother won't know.

For example, suppose you have 7 math problems for homework tonight. Then you say **double the number of math problems that we have for homework tonight and add 1**. If your classmate remembers that you both have 7 problems then she computes

$$2 \times 7 + 1 = 15$$

and has the secret!

This is called a **shared secret** and the same basic idea is used in the simplest encryption algorithms.

---

## Public Keys and Private Keys

---

In modern cryptography a **Key** is basically a long string of random numbers and letters. For example, the following string has 32 digits (in our base 10 decimal system). Remember though that the computer stores information in bits.

3048 0241 0019 1826 9786 2545 8912 9037

A **Public Key** is just what its name suggests – public. It is made available to everyone via a publicly accessible directory or repository.

A **Private Key** is just what its name suggests – private. Only the owner has the key. A Private Key can be shared as in our example.



The length of the Key determines the number of possibilities which in turn indicates how many steps it would take if one tried a Brute Force approach. We have already seen how to do this before. For example in a monoalphabetic cipher using our 26 letters there are  $26 \times 25 \times \dots \times 2 \times 1 = 26!$  combinations but with a polyalphabetic cipher using our 26 letters there are  $26 \times 26 \times \dots = 26^{26}$  possibilities.

However, just like memory on your computer, the Key length is described in terms of bits so we would like to know what actual length (in our decimal system) this corresponds to.

For example, if the Key length is 2 bits then we know that only 1,2,3 can be represented (as 1, 10, 11). Think of a bicycle lock so we have  $3 \times 2 \times 1 = 3! = 6$  possible keys.

If the Key length is 3 bits then looking at our table where we counted to 20 we see that only 1-7 can be represented so we have  $7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 7! = 5040$  possible keys.

If the Key length is 4 bits then looking at our table we see that the largest decimal number is 15 that can be represented so we have  $15!$  possible keys.

Decimal	Binary	Decimal	Binary	Decimal	Binary	Decimal	Binary
1	1	6	110	11	1011	16	10000
2	10	7	111	12	1100	17	10001
3	11	8	1000	13	1101	18	10010
4	100	9	1001	14	1110	19	10011
5	101	10	1010	15	1111	20	10100

What is the largest number we can represent with 5 digits?

We know that it takes 5 bits to represent 20 but we can still represent a larger number with 5 bits. One way to determine this number is to keep counting in binary but an easier way is to realize that the largest 5 bit number is **11111** and ask what decimal number this represents. Recall that to do this we write

$$11111 = 1 \cdot 2^0 + 1 \cdot 2^1 + 1 \cdot 2^2 + 1 \cdot 2^3 + 1 \cdot 2^4 = 1 + 2 + 4 + 8 + 16 = 31$$

So using a binary key of length 11111 means we can represent the numbers 1-31 so we have 31! combinations.

However, we would like to know a “rule of thumb” to convert a large binary number to decimal. We have the following formula for approximating this relationship.

If  $m$  is the number of binary digits in a number, then the number of decimal digits  $n$  is approximated by

$$n \approx 0.3m = 30\%(m)$$

For example, a 56-bit length Key is really a Key length of approximately  $.3 \times 56$  decimal digits or about 17 decimal digits.

How did we get this “30%” rule?

It turns out that you actually need to use logarithms to show this. However, for our needs we will just make a table comparing the number of binary digits to the number of base 10 digits in a number. As you can see from the table, 30% is a rough estimate (this actually comes from the fact that  $\log_{10} 2 = 0.301$

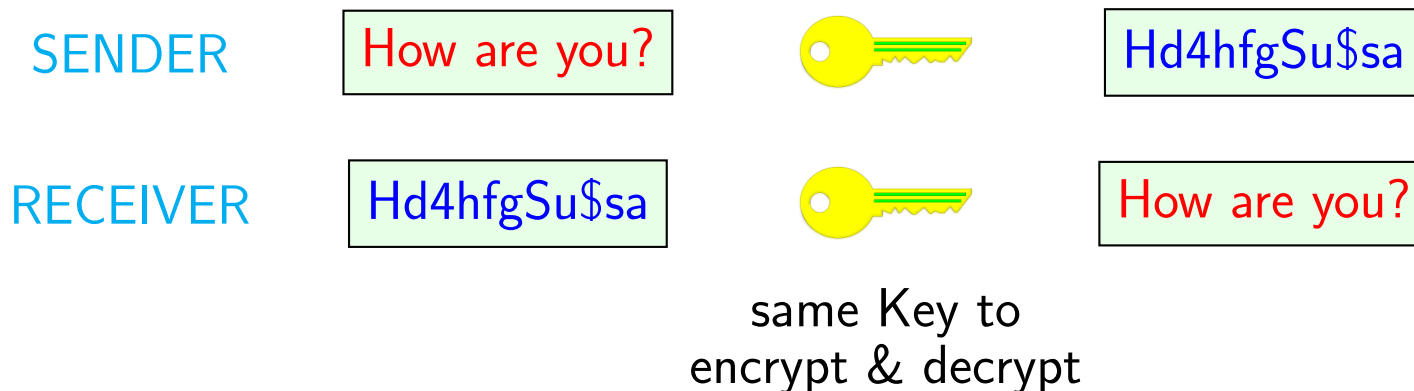
	# binary digits	# base 10 digits	Percent
$2^4 = 16$	5	2	40%
$2^5 = 32$	6	2	33%
$2^6 = 64$	7	2	28.5%
$2^7 = 128$	8	3	37.5%
$2^8 = 256$	9	3	33%
$2^9 = 512$	10	3	30%
$2^{10} = 1024$	11	4	36.4%
$2^{11} = 2048$	12	4	33%
$2^{12} = 4096$	13	4	30.8%
$2^{13} = 8192$	14	4	28.5%
$2^{14} = 16384$	15	5	33%

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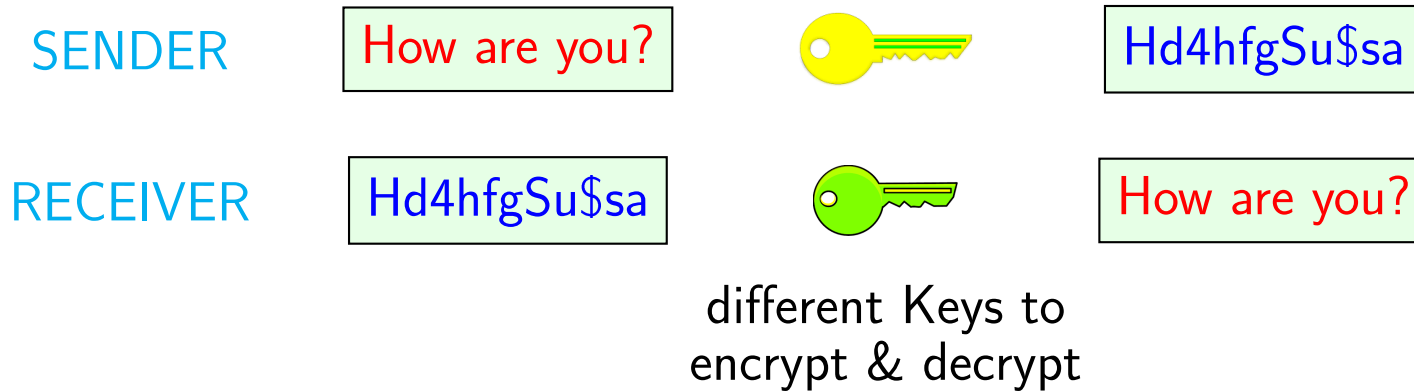
## Types of Modern Encryption

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- There are two basic types of encryption
  - Symmetric Encryption
  - Asymmetric or Public Key Encryption
- Symmetric encryption uses the **same Keys** for encrypting and decrypting.
- The ciphers we have considered so far (shift, simple substitution, Vigenere) all use symmetric encryption.



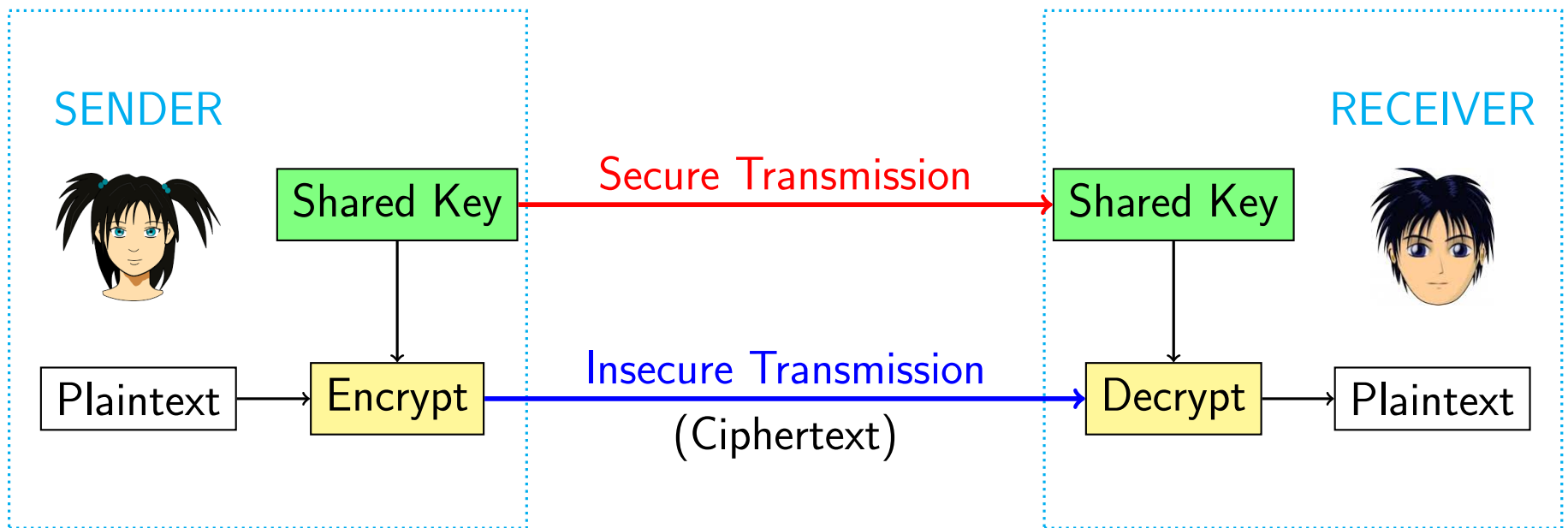
- Public Key (Asymmetric) encryption uses **different Keys** for encrypting and decrypting.



- Only the Green Key can decrypt what the Yellow Key encrypts.
- This takes sophisticated mathematical algorithms.



## Symmetric Encryption System



Both Sender and Receiver agree upon a Private Key which is shared in a secure manner. The Sender encrypts the message with the Private Key and because it is encrypted she can send it using an insecure method. The Receiver also has the Private Key and uses it to decrypt the message.

The simple example of telling your classmate a secret without your brother knowing is a Symmetric Encryption which used the Shared Private Key of knowing how many

math homework problems were assigned.

## Advantages of Symmetric Encryption Systems

- Same Key is used to encrypt the data and to decrypt the data so only one Key is needed
- Generally faster than Public Key encryption systems

## Disadvantages of Symmetric Encryption Systems

- Since both sender and receiver need the same Key it must be stored in a secure location that is accessible to both parties
- There needs to be a secure channel to transfer the Key
- Having to store the Key in a location, even if it is “secure”, lends itself to attacks.
- Keys should be changed regularly to prevent attacks

---

## Cast of Archetypal Characters in Cryptography

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- **Bob** and **Alice** are used for standard characters in cryptography. Saying “Alice wants to send Bob a message” is more natural to say than “Party A wants to send Party B a message”.
- **Eve** is an eavesdropper and is usually a passive attacker. She listens in on messages between Bob and Alice but can’t modify the message.
- **Mallory** is a malicious attacker who can modify messages, substitute their own messages, etc. Securing a system against Mallory requires much stronger security measures than against Eve.
- **Faythe** is a trusted advisor or intermediary such as a repository of key service. May be a machine or a human.
- **Craig** is a password cracker.
- There are many other archetypal characters; see Wikipedia, “Alice and Bob” entry or the article “Alice and Bob: IT’s inseparable couple”. Check it out to see if your name is a standard cryptography character!

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## A Simple Example of a Symmetric Key Encryption

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Alice wants to send a private message to Bob via postal mail. Unfortunately, she is convinced that the postman is reading the mail that she sends.



What can she do? She decides to put the message inside a lockbox, then mail the box to Bob. She buys a lockbox with two identical keys.



She doesn't trust her postman so she is afraid if she mails a key to Bob then he might make a copy. So she has to set up a meeting with Bob to give him a key but she only has to do this once, no matter how many messages she sends.



Alice then writes her message and locks it in the lockbox and mails it to Bob.



The mailman can see the lockbox but he can't open it. There is no way the postman can read the message so it is secure!

Bob can read the message because Alice and Bob each have a Key.

- This is a **symmetric encryption system** because the Key (in this case a physical key) is the same to encrypt (lock the lockbox) and decrypt (unlock the lockbox).
- The Key (here the physical object) only has to be securely sent once.
- This type of security was used in armored cars. At one location, items are loaded into the back of the van and an overseer locks the back with a key that he/she

keeps. An identical key has been given to the receiver of the goods. For security reasons the driver can't open the back so it can't be opened until it gets to its final destination.

But what if Bob lives in New Zealand?



---

## TimeLine for First Symmetric Encryption Key Algorithms

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May 1973	U.S. Government publishes a first request for a standard encryption algorithm
August 1974	U.S. Government publishes a second request for a standard encryption algorithm
March 1975	Data Encryption Standard (DES) published for review
November 1976	DES is approved as standard
1986	HBO begins using TV satellite scrambling system based on DES
June, 1997	Message encrypted with DES is publically broken
November 2001	Advanced Encryption Standard (AES) published
May 2002	AES becomes standard

DES (Data Encryption Standard)

1976, 56-bit Key length

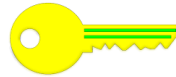
broken, 1997 (56 hours)

AES (Advanced Encryption Standard)

2001, 128-256 bit Key length

broken, 2009

Symmetric Key



Algorithms

Blowfish

1993, 448 bit Key length

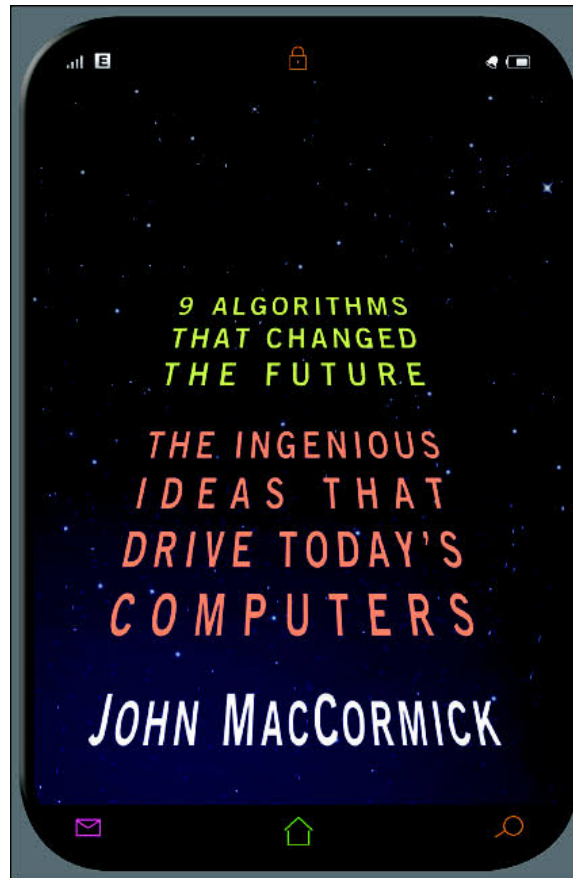
3 DES (Triple DES)

1998

International Data Encryption (IDEA)

1991, 128-bit Key length

Assignment: Read pages 44-59 of Algorithm 4



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## SOCRATIVE QUIZ - PartIII\_Quiz4

IUZGAZ34E

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1. Computers store information as \_\_\_\_\_.
  - a. binary digits
  - b. decimal numbers
  - c. bits
  - d. both (a) and (c)
  - e. both (b) and (c)
2. The binary number 10011 is larger than the binary number 10100.
3. The binary number 101011 is an even number in our decimal system.
4. All binary numbers which end in a 0 represent even numbers in our decimal system.
5. The binary number 101 represents what decimal number?
  - (a) 9

(b) 5

(c) 11

(d) 3

6. If a Key is represented by a 2 bit number, then xxx.

7. Approximately how many digits in base ten (our system) do you think it would take to represent an 128-bit Key?

a. 128

b. 64

c. 38

d. 30

e. 19

8. In a(n) \_\_\_\_\_ encryption system, the same Key is used by both the sender and receiver.

a. symmetric

b. asymmetric

c. Public-Key

d. both (a) and (c)

e. both (b) and (c)

9. In a(n) \_\_\_\_\_ encryption system two different Keys are used.

a. symmetric

b. asymmetric

c. Public-Key

d. both (a) and (c)

e. both (b) and (c)

10. A Substitution Cipher is an example of a \_\_\_\_\_ encryption system.

a. symmetric

b. asymmetric

c. Public-Key

d. both (a) and (c)

e. both (b) and (c)

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## Goals for Lecture

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1. To understand how Public Key Encryption differs from Symmetric Key Encryption
2. To understand the main idea behind Public Key Encryption by an analogy with mixing paint.
3. To introduce the concept of “clock arithmetic” and see how it is connected to modular arithmetic.
4. To get a glimpse of the mathematics behind Public Key Encryption by using modular arithmetic.

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## A Simple Example of Public Key Encryption

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In Public Key Encryption one Key is used to encrypt the message and another Key is used to decrypt it.

Alice wants to send a private message to Bob via postal mail. Unfortunately, she is convinced that the postman is reading the mail that she sends. The problem is that Bob lives in New Zealand and Alice lives in Florida so they can't easily meet.

Bob buys a padlock with a key. He keeps the key and mails the unlocked padlock to Alice.



Alice buys a simple lockbox that closes with a padlock. She writes her message, puts it in the lockbox and locks it with the padlock Bob sent her and mails it back to Bob.





The postman can't read the message because there is only one key to the padlock which Bob has.

When Bob gets the package he uses his key to open the lockbox and read the message.



In order for Alice to send another message to Bob, he has to mail the unlocked lockbox and the opened padlock back to Alice.

It is Public Key system (asymmetrical) because Bob's Private Key is the physical key to decrypt the message (i.e., unlock the box) and the encryption is done by the padlock which is the Public Key because it was sent by regular mail so anyone could duplicate it.

Of course this only works for sending messages one-way. For two-way messages they each would have to have their own padlock and keep the key for it.

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## Shared Secret Key

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- We know that in Public Key Encryption a different key is used to encrypt the message and another one to decrypt the message.
- In practice, Public Key Encryption typically uses a [Shared Secret Key](#). Remember in the example when you wanted to tell your classmate a secret without your younger sibling hearing it you had to find a way for your classmate to figure out the secret without you saying it.
- The problem is that both the sender and receiver must figure out the shared secret Key from some public information without actually transmitting it back and forth.
- For example, the first time you sign on to a new secure website to make a purchase, then your computer and the receiver must agree on a secret code before your sensitive information is sent.
- To understand how this is accomplished we first look at an analogy of paint mixing and then we see how it could be done in practice.

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## A Simple Analogy for Forming a Shared Secret - Paint Mixing

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**Goal:** Bob and Alice want to mix a secret paint color but Eve is eavesdropping and they don't want her to know their special color.

**Strategy:** Bob and Alice will mix the same color without Eve knowing and without directly sharing the formula for the new color. This formula will play the role of the **shared secret key**.

**Assumptions:**

1. Bob, Alice and Eve are in a room which has a curtained off area for Bob and another for Alice so that they can mix paint in private.
2. In the center of the room are paint pots of say 1,000,000 different colors.
3. There are several pots of each color of paint.

## Rules:

1. Neither Bob nor Alice can give a paint pot directly to the other person.
2. If information (i.e., a paint pot) is to be shared, then it must be placed in the center of the room. Multiple copies of that particular color must be shared.



EVE

Bob and Alice want to mix a secret paint color  
without Eve knowing

**Step 1.** Before beginning, several pots of a single color have been placed in Bob's private corner and several pots of another color have been placed in Alice's private corner. We refer to this as Bob's or Alice's **private** color of paint. We assume that this was done without the knowledge of anyone else.

Let's assume that Bob has **blue** and Alice has **crimson**. Since there are a million different colors, the likelihood of Bob and Alice having the same color is small.

Only Bob knows that his private color is blue.

Only Alice knows that her private color is crimson.

Eve doesn't know either Bob's or Alice's private color.



Bob and Alice each have a secret paint color

**Step 2.** Alice chooses a **public** color of paint.

Alice announces to the room that the public color is **pale yellow**.

So Alice, Bob and Eve know the public color, pale yellow.



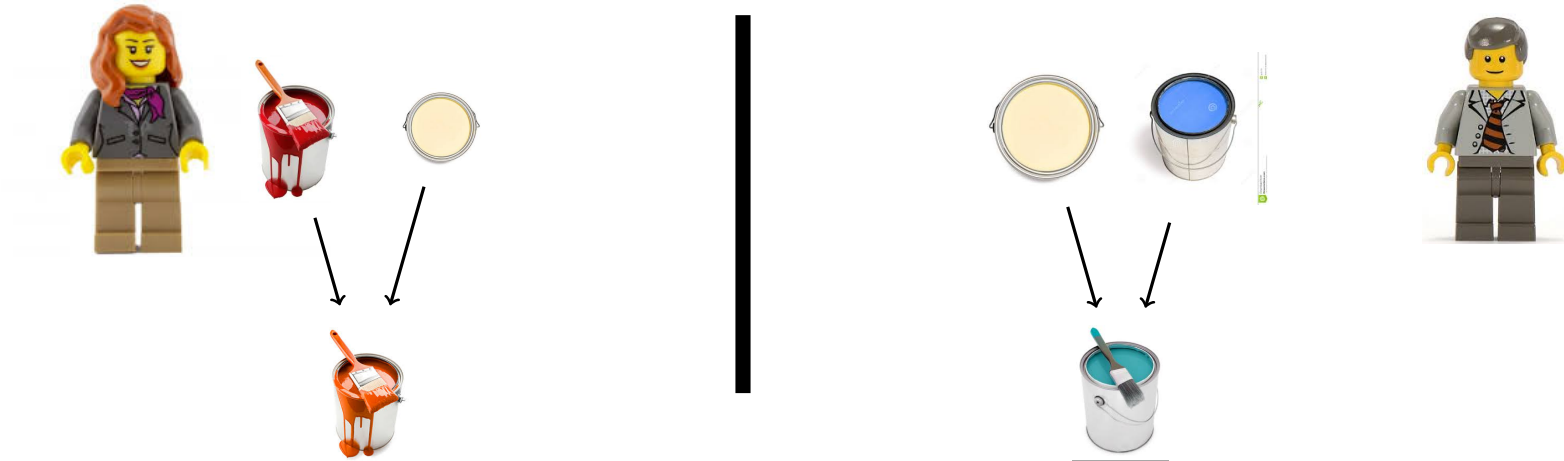
Everyone knows the public color is pale yellow



**Step 3.** Alice and Bob each get several paint pots of pale yellow paint. Behind their respective curtains they mix 1 pot of pale yellow with one pot of their private color. They repeat the process to get several **public-private** pots.

Alice now has paint pots that are half pale yellow and half crimson.

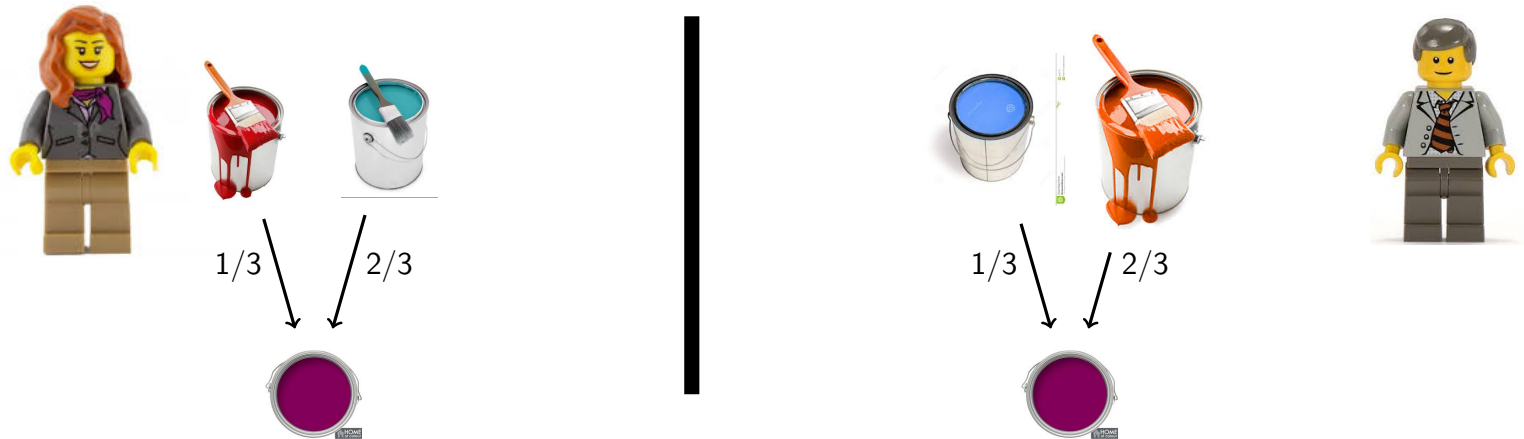
Bob now has paint pots that are half pale yellow and half blue.



Bob and Alice each mix a public-private color

**Step 4.** Alice and Bob place their public-private pots in the center of the room and take a pot of each other's back to their corners.

**Step 5** Alice mixes two parts of Bob's public-private pot with one part of her private color. Bob mixes two parts of Alice's public-private pot with one part of his private color.



Bob and Alice each have a paint that is  $1/3$  pale yellow,  $1/3$  crimson and  $1/3$  blue

- Alice has a pot of paint that is  $\frac{2}{3}$  Bob's public-private color ( equal parts blue and pale yellow  $\implies$  bluish green) and  $\frac{1}{3}$  of her private color (crimson) so her paint color is  $\frac{1}{3}$  pale yellow (public color) ,  $\frac{1}{3}$  blue (Bob's private color) and  $\frac{1}{3}$  crimson (her private color).
- Bob has a pot of paint that is  $\frac{2}{3}$  Alice's public-private color (equal parts crimson and pale yellow  $\implies$  orangish) and  $\frac{1}{3}$  of his private color (blue) so his paint color is  $\frac{1}{3}$  pale yellow (public color) ,  $\frac{1}{3}$  crimson (Alice's private color) and  $\frac{1}{3}$  blue (his private color).
- They both have the formula for a new paint color without broadcasting it.

Now we want to use the idea behind mixing paint and apply it to real-world applications.

In the real-world we use numbers instead of paint colors. Instead of mixing paints we perform operations on numbers.

Remember, when we mixed the paints together this process was irreversible so we need a mathematical process like this.

Clearly addition is reversible by using subtraction. That is, if we add 9 to a number, say 22 to get 31 then we can subtract 9 from 31 to get 22. This is because addition and subtraction are inverse operations.

Multiplication is reversible by using division. For example, if we multiply 12 by  $\frac{3}{4}$  to get 9 then we divide 9 by  $\frac{3}{4}$  (i.e., multiply by  $\frac{4}{3}$ ) to get 12. This is because multiplication and division are inverse operations.

We now want to look at a type of arithmetic that does not have an inverse. Mathematicians call this **modular arithmetic** but to explain it we will call it **clock arithmetic**. You will see that you have been doing modular arithmetic most of your life without realizing it!

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## Clock Arithmetic (a.k.a. Modular Arithmetic)

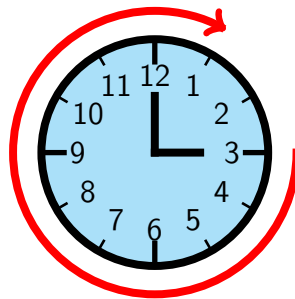
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Let's start with the typical clock with 12 numbers on it.

Suppose it is 3 o'clock and we want to know what time it will be in 10 hours. Clearly we don't say 13 o'clock. As we move around the clock (clockwise) adding 10 hours when we pass 12 we set the counter to 0. So instead of saying 13 o'clock we say 1 o'clock.

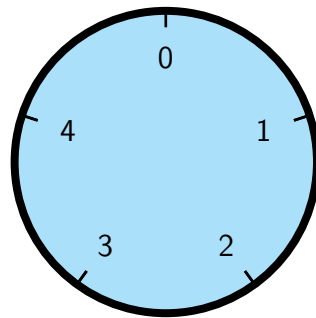
So using 12-hour clock arithmetic we say that  $3 + 10 = 1$ !

If there is a meeting that you know lasts 2 hours and it ends at 1 o'clock we know that it started at 11 o'clock so in 12-hour clock arithmetic we are saying  $1 - 2 = 11$ .



We use a 12 hour analog clock but we can do clock arithmetic in the same way even if we have a 5 hour or a 22 hour clock.

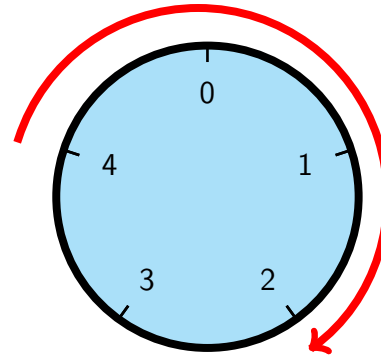
To make clock time correspond to modular arithmetic we adopt the convention that we use 0 instead of the last hour. For example, if we have a 5 hour clock the numbers on the dial are 0,1,2,3,4.



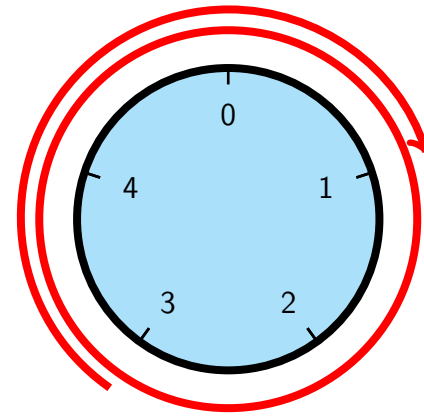
To do 5-hour clocktime arithmetic we simply mimic what we do on a 12 hour clock.

For example, if we want to add  $4 + 3$  we get 7 normally but in 5-hour clocktime we get 2 by adding 3 hours to 4 and since when we add 1 hour we get 0, then two more hours gets us to 2.

$4 + 3 = 2$  in  
5 hour clock arithmetic

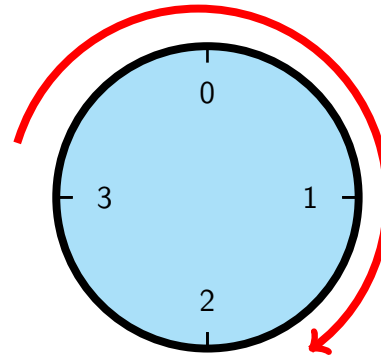


$3 + 8 = 1$  in  
5 hour clock arithmetic

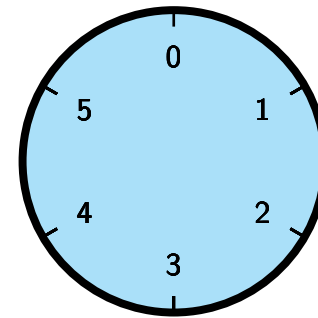


This approach works no matter how many hours we have on the clock

$3 + 3 = ?$  in  
4 hour clock arithmetic



$3 + 8 = ?$  in  
6 hour clock arithmetic





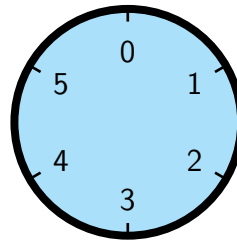
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## SOCRATIVE QUIZ - PartIII\_Practice\_Quiz?

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Use this 6-hour clock to do the following clock arithmetic.



1.  $5 + 2 = ?$

- (a) 1
- (b) 2
- (c) 3
- (d) 4
- (e) 5

2.  $3 + 4 = ?$

(a) 1

(b) 2

(c) 3

(d) 4

(e) 5

3.  $5 + 5 = ?$

(a) 1

(b) 2

(c) 3

(d) 4

(e) 5

4.  $5 + 11 = ?$

(a) 1

(b) 2

(c) 3

(d) 4

(e) 5

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## Relating Clock Arithmetic to Modular Arithmetic

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Now let's see how this is related to **modular arithmetic**. When you divide two integers you get an integer plus a **remainder**. For example

$$\frac{13}{5} = 2 \text{ remainder } 3$$

$$\frac{26}{11} = 2 \text{ remainder } 4$$

In **modular arithmetic** we are only interested in the remainder. The notation we use is

$$13 \text{ mod } 5 = 2 \qquad 26 \text{ mod } 11 = 4$$

So the number before the modulus ( **mod** ) is the numerator of the fraction and the number after it is the denominator and the answer is the remainder.

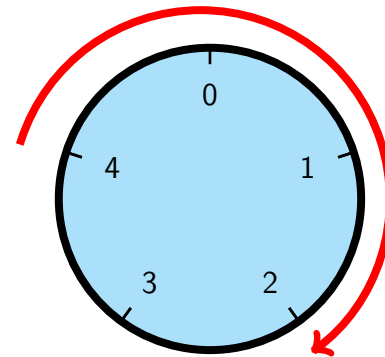
In the table below we give modular arithmetic using  $\text{mod } 5$ . Note how the remainder goes from 1 to 4, then 0 and repeats.

$1 \text{ mod } 5 = 1$	$6 \text{ mod } 5 = 1$	$11 \text{ mod } 5 = 1$	$16 \text{ mod } 5 = 1$
$2 \text{ mod } 5 = 2$	$7 \text{ mod } 5 = 2$	$12 \text{ mod } 5 = 2$	$17 \text{ mod } 5 = 2$
$3 \text{ mod } 5 = 3$	$8 \text{ mod } 5 = 3$	$13 \text{ mod } 5 = 3$	$18 \text{ mod } 5 = 3$
$4 \text{ mod } 5 = 4$	$9 \text{ mod } 5 = 4$	$14 \text{ mod } 5 = 4$	$19 \text{ mod } 5 = 4$
$5 \text{ mod } 5 = 0$	$10 \text{ mod } 5 = 0$	$15 \text{ mod } 5 = 0$	$20 \text{ mod } 5 = 0$

Now if we return to our example of doing 5 hour clock arithmetic we saw that  $4+3 = 2$  and notice that  $7 \text{ mod } 5 = 2$ . How do these relate?

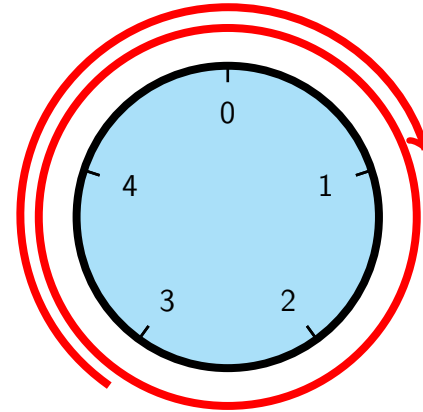
If we add 4 and 3 in the usual way we get 7 (the numerator in our fraction) and so since we are using a 5 hour clock this is our denominator and we write  $7 \text{ mod } 5$ .

$$\begin{aligned}
 &4 + 3 = 2 \text{ in} \\
 &5 \text{ hour clock arithmetic} \\
 &4 + 3 = 7, \frac{7}{5} = 1 \text{ remainder } 2 \\
 &7 \text{ mod } 5 = 2
 \end{aligned}$$



In our second example using a 5 hour clock we had  $3 + 8 = 1$ . If we add 3 and 8 we get 11 and  $11 \bmod 5 = 1$  because 5 goes into 11 twice with a remainder of 1.

$$\begin{aligned}
 &3 + 8 = 1 \text{ in} \\
 &5 \text{ hour clock arithmetic} \\
 &3 + 8 = 11, \frac{11}{5} = 2 \text{ remainder } 1 \\
 &11 \bmod 5 = 1
 \end{aligned}$$



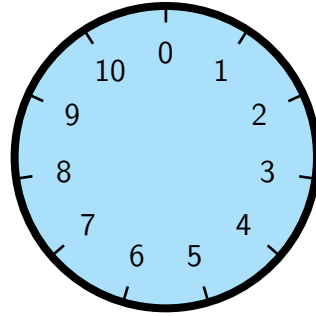
With modular arithmetic we can also do multiplication. For example

$$5 \times 4 = 20 \quad (5 \times 4) \bmod 5 = 0 \quad (5 \times 4) \bmod 6 = 3,$$

$$2 \times 2 \times 2 \times 2 = 2^4 = 16 \quad 2^4 \bmod 3 = 1 \quad 2^4 \bmod 11 = 5$$

In the second one we have used power notation, i.e., exponentiation.

For our example using Public Key encryption we will use a clock with 11 hours because it's easy to multiply 11 by integers.



For the calculations below we need to know multiples of 11 so we make a table for reference. The first column is  $n$  the number we are multiplying 11 by and the second column is the result.

### Multiplication of $n \times 11$

$n$	$11 \times n$	$n$	$11 \times n$	$n$	$11 \times n$	$n$	$11 \times n$	$n$	$11 \times n$
1	11	2	22	3	33	4	44	5	55
6	66	7	77	8	88	9	99	10	110
11	121	12	132	13	143	14	154	15	165
16	176	17	187	18	198	19	209	20	220
21	231	22	242	23	253	24	265	25	275

Now we can look at the table to immediately perform modular arithmetic where we use powers of integers. We have

$$213 \pmod{11} = 4$$

To do this we look at the table and find that 209 and 220 sandwich 213. Since  $11 \times 19 = 209$  and  $11 \times 20 = 220$  we know that 19 is the largest integer which divides 213 and the remainder is  $213 - 209 = 4$ .

Likewise

$$3^5 = 243 \implies 3^5 \pmod{11} = 1$$

because  $11 \times 22 = 242$  and  $11 \times 23 = 253$  so the largest integer dividing 243 is 22 and the remainder is  $243 - 242 = 1$ .



## SOCRATIVE QUIZ - PartIII\_Quiz15

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Use the table below to answer the following questions.

$n$	$11 \times n$	$n$	$11 \times n$	$n$	$11 \times n$	$n$	$11 \times n$	$n$	$11 \times n$
1	11	2	22	3	33	4	44	5	55
6	66	7	77	8	88	9	99	10	110
11	121	12	132	13	143	14	154	15	165
16	176	17	187	18	198	19	209	20	220
21	231	22	242	23	253	24	265	25	275

1. If  $9^2 = 81$ , what is  $9^2 \pmod{11}$ ?
2. If  $2^8 = 256$ , what is  $2^8 \pmod{11}$ ?
3. Let  $n$  be any positive integer. What is the largest value that  $2^n \pmod{11}$  can be?

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## SOCRATIVE QUIZ ANSWERS

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Use the table below to answer the following questions.

$n$	$11 \times n$	$n$	$11 \times n$	$n$	$11 \times n$	$n$	$11 \times n$	$n$	$11 \times n$
1	11	2	22	3	33	4	44	5	55
6	66	7	77	8	88	9	99	10	110
11	121	12	132	13	143	14	154	15	165
16	176	17	187	18	198	19	209	20	220
21	231	22	242	23	253	24	265	25	275

1. If  $9^2 = 81$ , what is  $9^2 \pmod{11}$ ?

Looking at the table we see that  $11 \times 7 = 77$  and  $11 \times 8 = 88$  so 7 is the largest integer that divides 81 and the remainder is  $81 - 77 = 4$  so  $9^2 \pmod{11} = 4$ .

2. If  $2^8 = 256$ , what is  $2^8 \pmod{11}$ ?

Looking at the table we see that  $11 \times 23 = 253$  and  $11 \times 24 = 265$  so 23 is the largest integer that divides 256 and the remainder is  $256 - 253 = 3$  so  $2^8 \pmod{11} = 3$ .

3. Let  $n$  be any positive integer. What is the largest value that  $2^n \pmod{11}$  can be? When we do modular arithmetic  $\pmod{11}$  the only values that are possible are  $0, 1, 2, 3, 4, \dots, 10$  so 10 is the largest value it can be. For example  $10 \pmod{11} = 10$  or  $21 \pmod{11} = 10$ .

We're almost ready to do the actual example. However, we will need results such as  $5^4 \pmod{11}$ ,  $9^7 \pmod{11}$  so instead of computing each of these separately, I'll precalculate some of them and put in a table. I have only included some terms that we will need in the examples. For homework you will be adding more; these are determined in exactly the same manner as the above examples.

$n$	1	2	3	4	5	6	7	8	9	10
$2^n$	2	4	8	5	10	9	7	3	6	1
$3^n$	3	9	5	4	1	3	9	5	4	1
$4^n$	4	5	9	3	1	4	5	9	3	1
$5^n$	5	3	4	9	1	5	3	4	9	1
$6^n$	6	3	7	9	10	5	8	4	2	1
$9^n$	9	4	3	5	1	9	4	3	5	1

Before we start the example, it's important to realize that even if we know that  $3^n \pmod{11} = 1$  we still can't figure out what  $n$  is. If we look at the second line of the table we see that  $n$  could be 5 or 10; continuing this pattern it could be 15, 20, 25, etc.

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## Public Key Encryption Example Using Modular Arithmetic

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**Goal:** To create a shared secret in such a way that the secret can't be obtained by someone observing the communication.

**Step 1.** Bob and Alice each choose a **Private Key**.

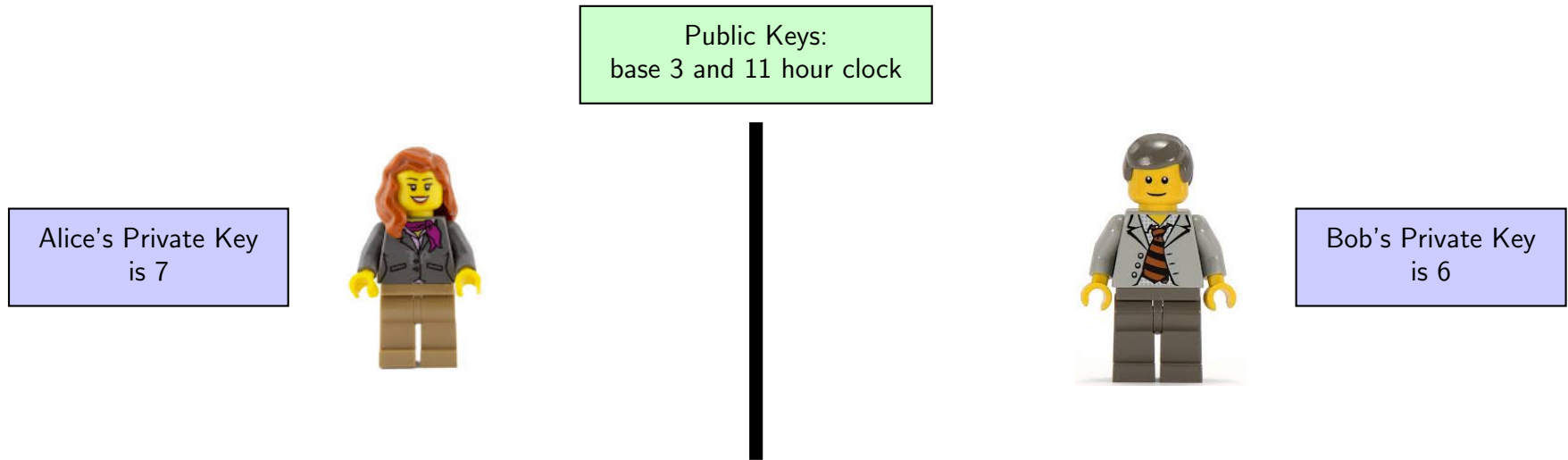
Alice's Private Key  
is 7



Bob's Private Key  
is 6



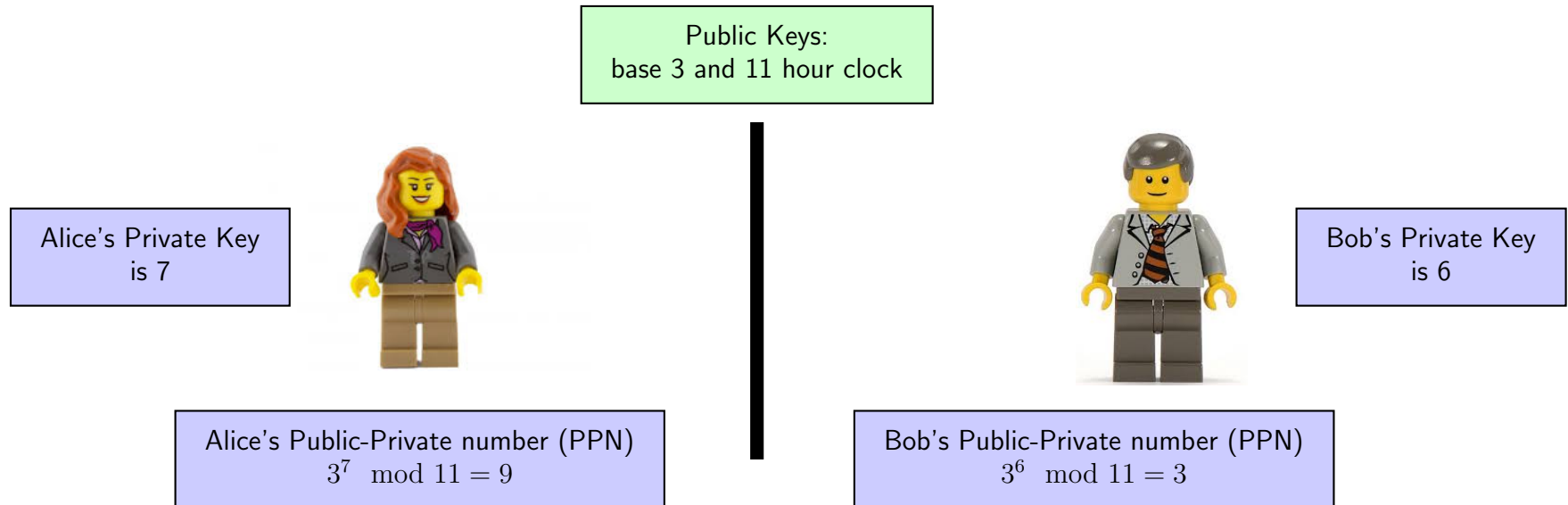
**Step 2.** Bob and Alice agree on **Public Keys**. In particular they choose a base (so we can use our table it should be 2, 3 or 6) and the number of hours in the clock; again so we can use the table we choose 11.



**Step 3** Alice and Bob each generate a number using their Private number, the public base and the 11 hour clock arithmetic. They use the following formula to get their Public-Private number (PPN)

$$\text{Alice: } (\text{public base})^{\text{Alice's Private No.}} \pmod{11}$$

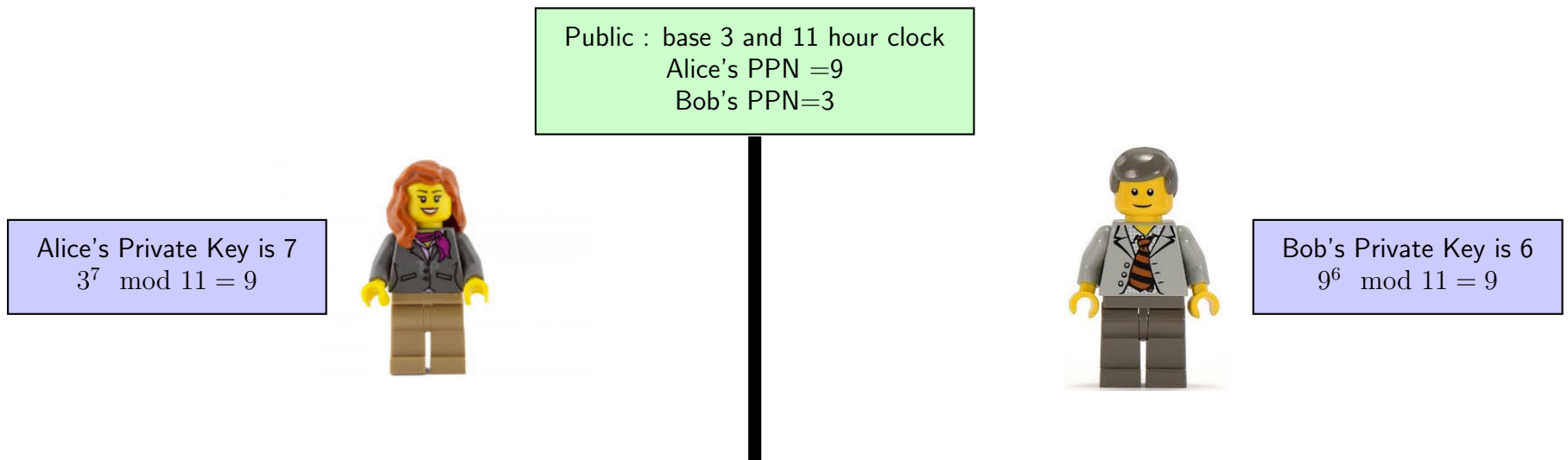
$$\text{Bob: } (\text{public base})^{\text{Bob's Private No.}} \pmod{11}$$



**Step 4** Bob and Alice exchange their Public-Private numbers. They then perform modular arithmetic to get the SAME number which is the **shared secret**. Alice and Bob use the following formulas

$$\text{Alice: } (\text{Bob's PPN})^{\text{Alice's private number}} \pmod{11}$$

$$\text{Bob: } (\text{Alice's PPN})^{\text{Bob's private number}} \pmod{11}$$





This works for any choice of numbers. Let's try a different set of Private numbers and base (but we will still use the 11 hour clock so we can use our table)

Step	Alice	Bob	Alice	Bob
Private #	8	2	4	9
Public base	5	5	2	2
PPN	$5^8 \bmod 11 = 4$	$5^2 \bmod 11 = 3$	$2^4 \bmod 11 = 5$	$2^9 \bmod 11 = 6$
Shared secret	$3^8 \bmod 11 = 5$	$4^2 \bmod 11 = 5$	$6^4 \bmod 11 = 9$	$5^9 \bmod 11 = 9$

Comments:

- Eve, an eavesdropper, knows the base, the number of hours in the clock and the two PPNs but she can't determine the secret number from these because she doesn't know Alice's or Bob's private number.
- In these examples we have chosen a small clock number. This is a problem because, e.g., when you are doing  $\bmod 11$  arithmetic then the possibilities for the shared secret are  $0, 1, 2, \dots, 10$  and so this is not very secure. In practice much larger numbers are used.

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## Public Key Encryption in Practice - Diffie-Hellman Algorithm

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- The goal of the algorithm is the same as in the example – a shared secret is generated between two parties in such a way that it can't be seen by an observer. Even someone analyzing the communication at a later date can't figure out the secret.
- Suppose you sign on to a secure website (https) such as your electric company to pay your monthly bill.
- Your computer and the internet server create a shared secret in much the same way as in our example.

## How is the algorithm different from our example?

1. The clock size now is hundreds of digits in length so that the number of possibilities for private numbers is enormous.
2. It also turns out that the clock size should be a prime number. What is this? A prime number is only divisible by itself and 1. For example,

1, 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43

are the first 15 prime numbers. No even number is prime except 2.

3. Another property of this algorithm is that the public base that is chosen must be related to the clock in a particular way. Let's go back to our table where we computed  $b^n \pmod{11}$  for different bases  $b$ . If we compare the line in the table for  $2^n$  and  $3^n$  we notice that there is a difference. The line for  $2^n$  contains ALL the numbers from 1 to 10 whereas the line for  $3^n$  is missing 2, 6, 8, and 10. The second requirement of the algorithm is that the base is chosen so that all possible numbers on the clock (except 0) occur when we perform modular arithmetic with this base.

## SOCRATIVE QUIZ - PartIII\_Quiz16

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1.  $5 + 4 = ?$  in 7 hour clock arithmetic
  - a. 1
  - b. 2
  - c. 3
  - d. 4
2.  $5^2 \bmod 4 = ?$ 
  - a. 1
  - b. 2
  - c. 3
  - d. 4
3. One commonly used Public Key Encryption using a shared secret Key is
  - a. AES

- b. DES
- c. Blowfish
- d. Diffie-Hellman

4. When using a Public Key Encryption algorithm with a Shared Secret Key then
- a. the shared secret Key is transmitted to sender and receiver via a secure transmission
  - b. the sender and receiver both use public information and a mathematical computation to compute the Shared Secret Key
  - c. the sender uses public information and a mathematical computation to compute the Shared Secret Key and transmits to the receiver via a secure transmission
  - d. the receiver uses public information and a mathematical computation to compute the Shared Secret Key and transmits to the sender via a secure transmission

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## DIGITAL SIGNATURES

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### Goals for this lecture:

- To understand what a digital signature is and how the goal of it differs from Public Key Encryption.
- To see how digital signatures are used in practice.
- To see how modular arithmetic can be used in a manner similar to Public Key encryption to solve this problem

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## What are Digital Signatures used for?

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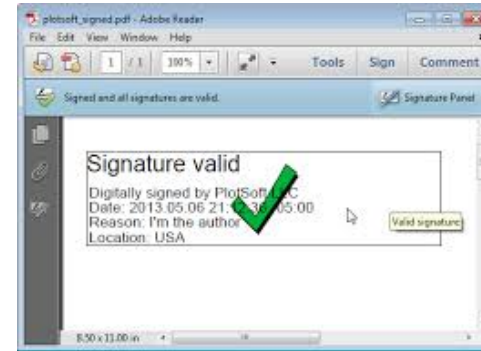
The phrase “digital signature” is a bit of an oxymoron. Clearly we use the term “signature” as something we can read but which can’t be forged. However “digital” literally means “a string of digits” and we know that if we can read the string of digits then we can copy them.

The difference in digital signatures and other cryptography we have looked at is that the goal is **NOT secrecy but authenticity**. That is, we want to know that the digital signature is not forged.

The obvious use of digital signatures is for things that paper signatures are used for such as signing an apartment lease from long distance.

We will see that they have another important use too.

Sending hard copies (i.e., the physical item) of documents (such as contracts, etc.) back and forth between organizations takes a lot of time. A time saving device is to use digital signatures.



The receiver of the document needs to be sure that the signature was done by the sender and no one else, i.e., it needs to be **verifiable**. The receiver of the document needs to be sure that the signature was **not forged**.



Using a **mobile ID** you can sign documents or transfer money with the touch of your finger.



In the European Union a digital signature carries the same weight as a physical signature.

Studies have shown that using digital signatures can save up to a week's worth of time for each working age adult!

A study in Estonia (population 1.3 million ) has shown that using digital signatures saves a stack of paper the height of the Eiffel Tower (984 ft) every month!

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## How are Paper Signatures Verified?

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TITLE OF ACCOUNT	
 <b>Southeast Bank Ltd.</b>	
_____ BRANCH _____ 200	
<small>I/We the undersigned, request you to open a Foreign Currency Account in US Dollar/£/Pound/Euro in my/our name (s). I/We agree to comply with and be bound by the rules and regulations of the Bank relative to this Account and any amendments thereto. Declarations are on the reverse side.</small>	
FULL NAME	SIGNATURE
A/C HOLDER :	
NOMINEE :	
SPECIAL INSTRUCTION :	Introduced by
	Verified By
	Approved by

Dep-07

Suppose you write a check for \$1000 and your bank realizes that this is a large amount and wants to verify that you actually wrote it and that it is not forged. When you opened your account, the bank made you sign a signature card which they keep on file. The bank employee can physically go to the card and check your signature on the check against the one on file.

If someone has a document with your name signed on it and you claim that you didn't sign it, then prior copies of your signature must be compared with the one on the document to determine if it is a valid signature.

How can we mimic this procedure with a digital signature?

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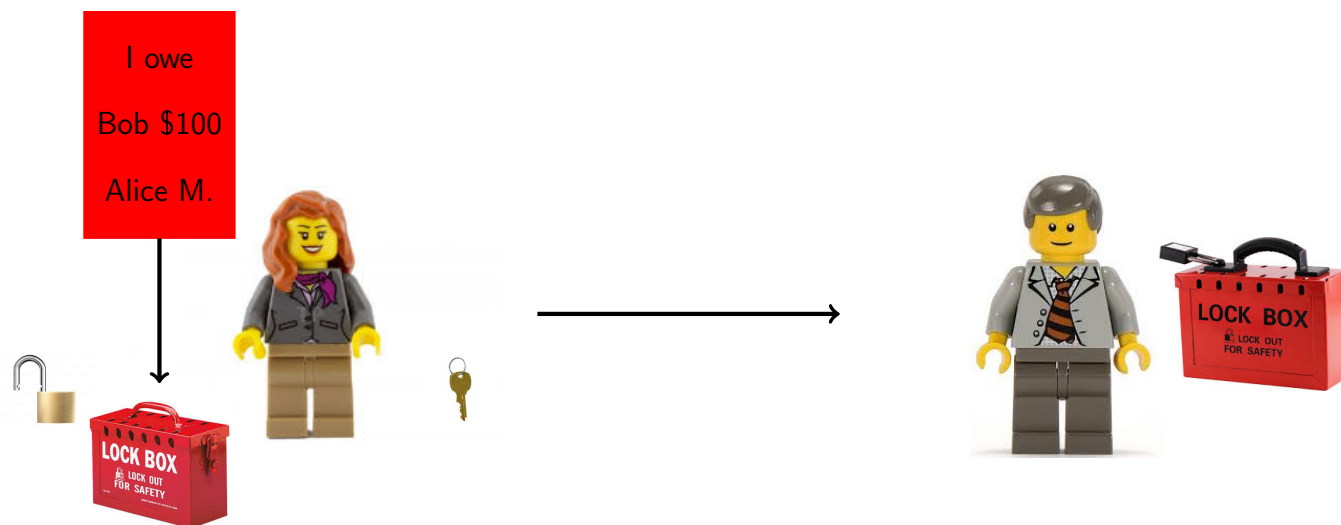
## Lockbox Example

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To understand how to verify a digital signature we first look at an example which uses physical objects instead of numbers and mathematics just like we did for Public Key Encryption.

Suppose Alice borrows \$100 from Bob. Being fiscally responsible, Bob asks Alice for an IOU. However, Bob is always losing things so he doesn't want a slip of paper.

Alice buys a lockbox and only she has a key. She write the IOU and puts it inside the lockbox and keeps the key. She gives the lockbox to Bob which is big enough that he won't lose it.



Several months pass and Bob asks Alice if she can repay the \$100. If Alice says “I don’t remember borrowing money from you,” Bob can produce the lockbox and ask Alice to give him the key. He then opens it and shows her the IOU.

The “signature” is verified since Alice is the only one with the key. Neither Bob nor anyone else could have opened the lockbox and forged the IOU.

## What can go wrong?

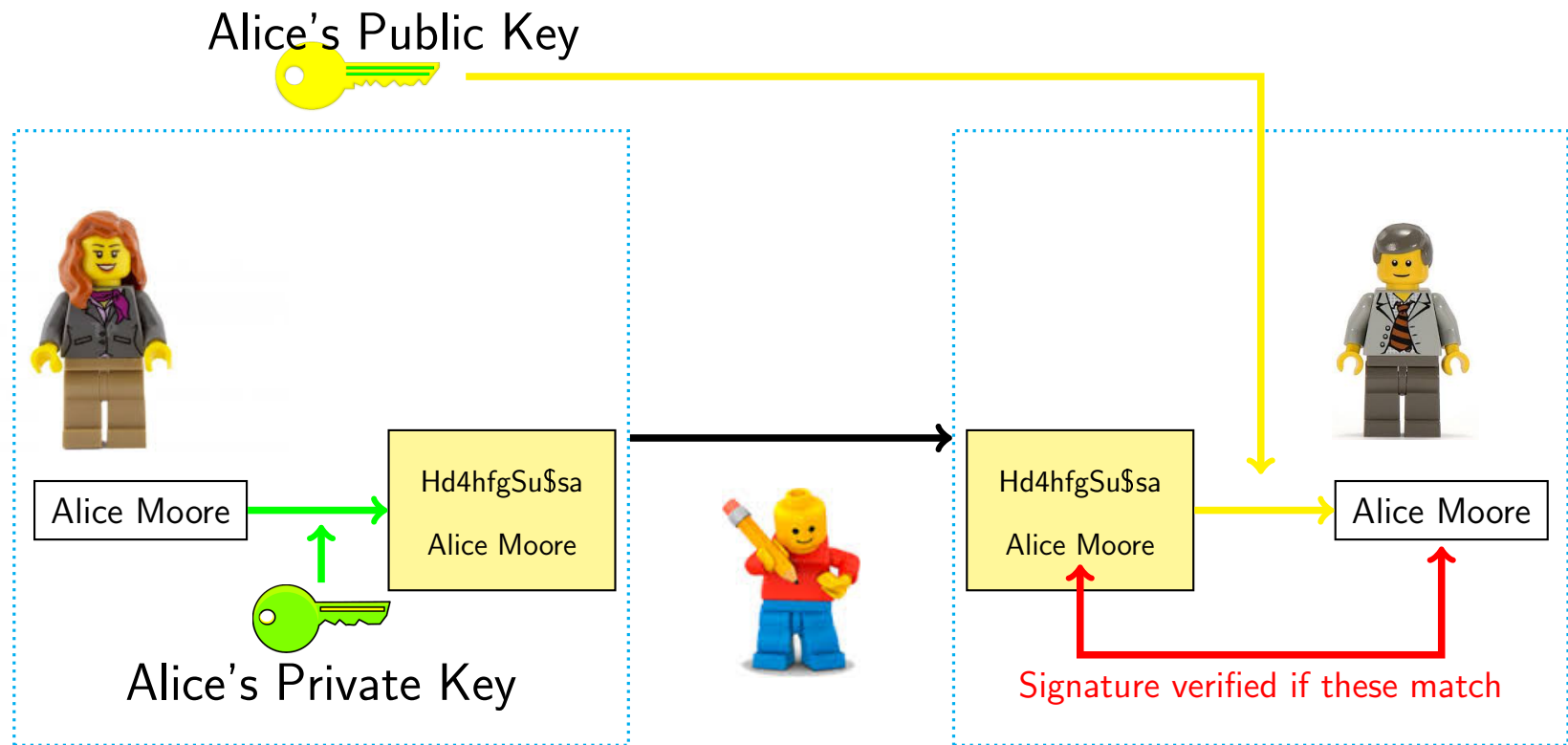
1. Alice could put any piece of paper (or nothing) into the lockbox instead of the IOU.
2. Alice could say that she lost the key.
3. Alice could give Bob the wrong key and when the lockbox wouldn't open she could respond "it isn't my lockbox."

To alleviate these problems, Bob could designate a trusted third party such as a bank. Alice would sign the IOU in the presence of the third party, give them the key for safe keeping and give Bob the lockbox. To verify check signatures the bank kept a signature card but now it will keep the physical key.

Now when Bob wants to see the IOU, he goes to the bank, gets the key and opens the lockbox in front of witnesses. The fact that the lockbox opens with Alice's key (which the bank is holding) authenticates the document inside.

To make this approach practical using computers we use numbers (hence the adjective “digital”) in place of lockboxes and mathematical operations instead of physical keys just like we did in Public Key Encryption.

Suppose now Alice wants to sign a document and send it to Bob. It does not need to be sent securely but Bob needs to know that it was Alice that signed the document. How can we do this?



The procedure is almost the reverse of Public Key Encryption. Recall that in Public Key Encryption the goal was for Alice to send a message securely to Bob. Now Alice wants to send a document with a digital signature to Bob but she doesn't want to see it. The goal is that the receiver (Bob) can **verify** it, i.e., know that it was sent by Alice and not by someone else.

In practice, the lockboxes will be represented by numbers and the locking/unlocking procedures will use **modular arithmetic** (i.e., clock arithmetic) as we did in Public Key Encryption. Also we will use a clock with 11 hours because it is so easy to multiply numbers by 11.

In Public Key encryption we used addition when using modular arithmetic but here we use multiplication; it works the same way though.

Let's review how to do this before we proceed. Remember that when using modular arithmetic all we are concerned about is the **remainder**.

Since we are using an 11 hour clock all of our expression will be of the form  $m \times n \bmod 11$  where  $m, n$  are integers. For example,

$$4 \times 3 \pmod{11} \rightarrow 12 \pmod{11} = 1$$

because when we divide 12 by 11 we get 1 with **remainder = 1**. Also

$$9 \times 3 \pmod{11} \rightarrow 27 \pmod{11} = 5$$

since  $27/11 = 2$  **remainder 5** and

$$5 \times 6 \pmod{11} \rightarrow 30 \pmod{11} = 8$$

since  $30/11 = 2$  **remainder 8**.

When you use software like **ADOBE** to digitally sign a document you simply type your name. Since computers prefer numbers, the first thing that happens is that your name is changed into a string of numbers like **37165900872144578**. Now remember that computers use binary numbers but for our purposes we won't complicate the discussion by using binary numbers.



To make things easy we start by assuming Alice's name can be transformed using only a single digit. Once we understand the approach we can see how to do the case when it is represented with multiple digits.

Assume Alice's signature is represented by the digit 3.

Alice selects a Public Key which in our case is the clock size of 11 and a number less than 11; here we choose 2.

Alice must have a Private Key to encrypt her digital signature of 3. To do this, Alice chooses a number less than 11 (her Private Key), say 6 and she uses multiplicative modular arithmetic to get

$$3 \times 6 = 18 \implies 18 \bmod 11 = 7$$

So Alice's digital signature is encrypted as 7. This is sent to Bob but anyone can see it. In addition Alice sends her digital signature of 3.

How does Bob unlock the signature and know that only Alice has sent it?

Because he unlocks the encrypted signature with her Public Key. How does he do

this? He takes the encrypted signature (7) and uses the Public Key of 2 to perform multiplicative modular arithmetic

$$7 \times 2 = 14 \implies 14 \bmod 11 = 3$$

Now he compares this with the original signature 3 which Alice has also sent unencrypted. If they match, then he knows that the encrypted document that Alice sent is not forged.

This verification will work no matter what Alice's original digital signature is, as illustrated in the table below.

Digital Signature	Private Key	Encryption	Public Key	Decryption
1	6	$1 \times 6 \pmod{11} = 6$	2	$6 \times 2 \pmod{11} = 1$
3	6	$3 \times 6 \pmod{18} = 7$	2	$7 \times 2 \pmod{14} = 3$
4	6	$4 \times 6 \pmod{11} = 2$	2	$2 \times 2 \pmod{11} = 4$
5	6	$5 \times 6 \pmod{11} = 8$	2	$8 \times 2 \pmod{11} = 5$
9	6	$9 \times 6 \pmod{11} = 10$	2	$10 \times 2 \pmod{11} = 9$

Of course the **Private Key** and the **Public Key** have to be related. For example, if Alice's digital signature is represented by 3 (which everyone knows) but if someone intercepts this and encrypts it with 5 instead of 6 as Alice did then Bob will find that the signature is a forgery because

$$3 \times 5 = 15 \implies 15 \pmod{11} = 4$$

is the encrypted signature Bob receives and decrypting it with Alice's Public Key (2) gives

$$4 \times 2 = 8 \implies 8 \pmod{11} = 8$$

and  $3 \neq 8$ .

How did we know that 2 was the Key which decrypted Alice's encryption Key of 6?

It turns out that the basis is an algorithm that was written over 2000 years ago!

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## Euclid's Algorithm

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Euclid's Algorithm (Greek mathematician around 300 BC) is a clever way of computing the **greatest common divisor (GCD or gcd)** of two integers; i.e., finding the largest integer which divides both numbers evenly (without a remainder). This is probably the oldest algorithm still in use.

The original importance of this idea was probably for measurements in geometry. For example, suppose you have two rulers of length 18" and 24" and you wanted to find the length of a third ruler which is as long as possible but that you can use it as a scale for the other two rulers. For example, you could have rulers of length 1", 2", 3", and 6" so the longest is 6". This is equivalent to finding the  $\text{gcd}(18,24)$ .

As another example, suppose we have the integers 16 and 24 and we want to find the largest integer which divides both evenly. Clearly the gcd has to be  $\leq 16$ , the smaller of the two integers. Determining the gcd in this case is easy because we know that  $8 \times 2 = 16$ ,  $8 \times 3 = 24$  and no integer larger than 8 divides them both so the gcd of 16,24 is 8.

Before we look at Euclid's algorithm we see what a Brute Force algorithm for finding

the gcd of two numbers would look like.

To determine  $\gcd(16,24)$ , we first note that the largest the gcd can be is 16 so the first guess is 16 and if that doesn't work we try 15, then 14, etc. When we find the first integer that divides both, then this is the answer because we want the largest such integer. To check to see if an integer divides 16 or 24 evenly we can perform modular arithmetic because all we have to do is check to see if the remainder is zero.

Why don't we start at 1 and increase the guess by 1 each time?

Brute Force Algorithm for finding  $\gcd(16,24)$

**Step 1.** Set  $\text{gcd} = 16$

**Step 2.** Determine remainder  $R_1 = 16 \bmod \text{gcd}$ .

**Step 3.** If remainder  $R_1 \neq 0$  then set  $\text{gcd} = \text{gcd} - 1$  and go to Step 2

If remainder  $R_1 = 0$  then determine new remainder  $R_2 = 24 \bmod \text{gcd}$  and continue.

**Step 4.** If  $R_2 = 0$ , then stop.

Else if  $R_2 \neq 0$  then  $\text{gcd} = \text{gcd} - 1$  and go to Step 2.

Loop	Current guess for gcd	$R_1$	Action	$R_2$	Action
1	16	$16 \bmod 16 = 0$	go to Step 3	$24 \bmod 16 = 8$	go to Step 2
2	15	$16 \bmod 15 = 1$	reduce gcd & go to Step 2		
3	14	$16 \bmod 14 = 2$	reduce gcd & go to Step 2		
4	13	$16 \bmod 13 = 3$	reduce gcd & go to Step 2		
5	12	$16 \bmod 12 = 4$	reduce gcd & go to Step 2		
6	11	$16 \bmod 11 = 5$	reduce gcd & go to Step 2		
7	10	$16 \bmod 10 = 6$	reduce gcd & go to Step 2		
8	9	$16 \bmod 9 = 7$	reduce gcd & go to Step 2		
9	8	$16 \bmod 8 = 0$	go to Step 3	$24 \bmod 8 = 0$	Found GCD

To generalize this algorithm for any two integers we have the following.

**Brute Force Algorithm for finding  $\text{gcd}(a, b)$  where  $a < b$**

**Step 1.** Set  $\text{gcd} = a$

**Step 2.** Determine remainder  $R_1 = a \bmod \text{gcd}$ .

**Step 3.** If remainder  $R_1 \neq 0$  then set  $\text{gcd} = \text{gcd} - 1$  and go to Step 2.

If remainder  $R_1 = 0$  then determine new remainder  $R_2 = 24 \bmod \text{gcd}$  and continue.

Step 4. If  $R_2 = 0$ , then stop.

Else if  $R_2 \neq 0$  then  $\text{gcd} = \text{gcd} - 1$  and go to Step 2.

Now it only took 9 steps using the Brute Force approach to find  $\text{gcd}(16,24)$  but what if our numbers were much larger such as finding the  $\text{gcd}(105,252)$ ? It turns out that the  $\text{gcd}$  is 21 so we have to do  $105-21=84$  recursive loops to get the result. Note that  $105=21 \times 5$  and  $252=21 \times 12$ .

It turns out that  $17 = \text{gcd}(6409,42823)$  we have to do  $6409-17=6,392$  recursive loops. . But if I wanted to find  $\text{gcd}(6409,12818)$  we see by observation that we get the answer on the first step of the Brute Force algorithm because  $\text{gcd}(6409,12818)=6409$ . So all we can say about the number of recursive loops that it takes for the Brute Force approach is that it is  $\leq$  the smaller of the two numbers.

Now let's see how Euclid's algorithm works. The basis for the algorithm is that if we have an integer  $n$  that divides both  $a$  and  $b$  then it also divides  $a - b$ . For example,  $5 = \text{gcd}(15,100)$  but 5 also divides  $100-15=85$  and also  $5 = \text{gcd}(15,85)$  so instead of finding  $\text{gcd}(15,100)$  we could find  $\text{gcd}(15,85)$ . However it is also true that 5 divides  $100 - (2 \times 15)=70$  so we could find  $\text{gcd}(15,70)$ . Continuing in this manner we see that  $100 - (3 \times 15)=55$  so we could find  $\text{gcd}(15,55)$  and  $100 - (4 \times 15)=40$  so we could find



$\gcd(15,40)$ ;  $100 - (5 \times 15) = 25$  so we could find  $\gcd(15,25)$  and finally  $100 - (6 \times 15) = 10$  so we could find  $\gcd(10,15)$ . It will be quicker to find the gcd of small integers so the first step of Euclid's algorithm is to reduce the problem of finding  $\gcd(15,100)$  to finding  $\gcd(10,15)$ . The way we do this

$$\frac{100}{15} = 6 \quad \text{remainder} = 10 \implies 100 = 6(15) + \frac{10}{15}$$

and we know that the remainder can be found by using modular arithmetic

$$10 = 100 \pmod{15}$$

.

Now what do we do? We want to find the  $\gcd(10,15)$  so we write

$$\frac{15}{10} = 1 \quad \text{remainder} = 5 \implies 15 = 10 + \frac{5}{10} \implies 5 = 15 \pmod{10}$$

so we have reduced our problem to determining  $\gcd(5,10)$  and so we write

$$\frac{10}{5} = 2 \quad \text{remainder} = 0 \implies 10 = 2 \times 5 \implies 0 = 10 \pmod{5}$$

and the answer is 5, the last number we divided by. We know we have the answer because our remainder is 0.

Let's do one more example before we write out the algorithm.

To find the  $\gcd(105,252)$  we have divide to 252 by 105 and find the remainder

$$\frac{252}{105} = 2 \quad \text{remainder} = 42 \implies 42 = 252 \pmod{105}$$

which gives us that  $\gcd(105,252)=\gcd(105,42)$ . Now we have

$$\frac{105}{42} = 2 \quad \text{remainder} = 21 \implies 21 = 105 \pmod{42}$$

and we have reduced the problem to find  $\gcd(21,42)$ . We have

$$\frac{42}{21} = 2 \quad \text{remainder} = 0 \implies 0 = 42 \pmod{21}$$

Note that in each case the remainder has to be less than the number you are dividing by.

We are done because our remainder is 0 so the gcd is the last number we divided by,

i.e.,  $\text{gcd}(105,252) = 21$ . This was found by performing 3 steps.

1. calculate  $252 \bmod 105$  (which is 42)
2. calculate  $105 \bmod 42$  (which is 21)
3. calculate  $42 \bmod 21$  (which is 0)

So what would the steps be if we wanted to find  $\text{gcd}(a,b)$  where  $a < b$

1. calculate  $c = b \bmod a$
2. if  $c \neq 0$  calculate  $d = a \bmod c$
3. if  $d \neq 0$  calculate  $e = c \bmod d$
4. etc.

Here is the pseudo-code for Euclid's algorithm.

Euclid's Algorithm for finding  $\text{gcd}(a, b)$  where  $a < b$

**Step 1.** Calculate  $c = b \bmod a$

**Step 2.** If  $c = 0$  , set  $\text{gcd}=a$  and stop.

If  $c \neq 0$  set  $b=a$  and  $a = c$ ; go to Step 1.

Let' s make a table like we did for the Brute Force algorithm for determining  $\text{gcd}(105,252)$ .

Loop	c	Action at Step 2
1		$42=252 \text{ mod } 105$ set $a=42, b=105$ & go to Step 1
2	$21 = 105 \text{ mod } 42$	set $a=21, b=42$ & go to Step 1
3	$0 = 42 \text{ mod } 21$	done; set $\text{gcd} = 21$

Now let's look at Euclid's algorithm for determining  $\text{gcd}(6409,42823)$  which took over 6000 loops for the Brute Force approach.

Loop	c	Justification for c	Action at Step 2
1	$4369=42823 \text{ mod } 6409$	$42823/6409 = 6 + 4369/6409$	set $a=4369, b=6409$ & go to Step 1
2	$2040 = 6409 \text{ mod } 4369$	$6409/4369=1 + 2040/4369$	set $a=2040, b=4369$ & go to Step 1
3	$289 = 4369 \text{ mod } 2040$	$4369/2040=2 + 289/2040$	set $a=289, b=2040$ & go to Step 1
4	$17 = 2040 \text{ mod } 289$	$2040/289= 7+17/289$	set $a=17, b=289$ & go to Step 1
5	$0 = 289 \text{ mod } 17$	$289/17 = 17$	stop, set $\text{gcd} = 17$

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## Socratic Practice Quiz xx IUK

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1. To determine the  $\text{gcd}(21,49)$  by the Brute Force method you start at 2 and increase the number until you find the first integer that divides both 21 and 49 evenly and that is your greatest common divisor.
2. In the first recursive loop of Euclid's algorithm for determining the  $\text{gcd}(21,49)$  one rewrites this problem in terms of
  - (a)  $\text{gcd}(21,70)$
  - (b)  $\text{gcd}(21,21)$
  - (c)  $\text{gcd}(21,28)$
3. In the second recursive loop of Euclid's algorithm for determining the  $\text{gcd}(21,49)$  one rewrites this problem in terms of
  - (a)  $\text{gcd}(7,21)$
  - (b)  $\text{gcd}(21,21)$
  - (c)  $\text{gcd}(7,28)$
4. Euclid's algorithm for determining  $\text{gcd}(a,b)$  terminates when

Now that we have seen Euclid's algorithm for determining the greatest common divisor of two integers, we want to see how this helps us to find **what relationship the Public and Private Keys must have.**

Recall that in our example, Alice chose a Private Key of 6 and we said that this means the Public Key (for an 11 hour clock) had to be 2. You can actually swap these around and have the Public Key of 6 and the Private Key of 2 and it still works because they are related in the same way.

Now what we want is to find the Public Key for an 11 hour clock that undoes the result of her Private Key of 6.

Lets go back to the example where we found that  $21 = \gcd(105,252)$ . What is someone asked you to find numbers  $x,y$  such that

$$21 = 105x + 252y$$

The sequence of equations that we found were

$$42 = 252 - (2 \times 105)$$

$$21 = 105 - (2 \times 42)$$

We want to use these to

Now using the first equation, the last equation can be written as

$$21 = 105 - (2 \times 42) = 105 - \left[ 2 \times \left( 252 - (2 \times 105) \right) \right]$$

Combining terms we get

$$21 = 105 - \left[ 2 \times \left( 252 - (2 \times 105) \right) \right] = 105 \times [1 + 4] + 252 \times [-2]$$

$$21 = 105 \times (5) + 252 \times (-2)$$

so  $x=5$  and  $y = -2$ .

Now what we want is to find the Public Key for an 11 hour clock that undoes the result of her Private Key of 6. We want to know the following Her digital signature represented by 3 is encrypted using the Private Key of 6 as

$$(3 \times 6) \pmod{11} = 7 \implies 7 = 18 - 11 \times 1$$

We want to know  $n$  such that

$$(n \times 7) \pmod{11} = 3$$

so we are really asking what is the inverse of 18 mod 11, i.e., what operation reverses

this process so that you get 3 back. This is equivalent to finding an  $n$  such that

$$(n \times 7) - 11k = 3$$

For some value of  $k > 0$ . By inspection  $n = 1$  doesn't work because we have  $7 - 11k \neq 3$  for any  $k$ . We can try  $n = 2$  to get  $14 - 11k = 3$  and this holds for  $k = 1$ .

Let's use Euclid's algorithm to find  $\gcd(11,18)$  which we know it is 1 since 11 is a prime number.

$$7 = 18 \pmod{11} \implies 18 - 11 = 7$$

$$4 = 11 \pmod{7} \implies 11 - 7 = 4$$

$$3 = 7 \pmod{4} \implies 7 - 4 = 3$$

$$1 = 4 \pmod{3}$$

Now we write 3 as a combination of 11 and 7 to get

$$3 = 7 - 4 = 7 - (11 - 7) = 2 \times 7 - 11$$

so we have found  $n = 2$ .



Let's try this with another choice of the Private Key, say 4. We use 4 to encrypt her digital signature which we now take as 9 to get

$$9 \times 4 \pmod{11} = 3 \quad \text{since} \quad \frac{27}{11} = 2 \times 11 + \frac{5}{11}$$

Now we want to find  $n$  such that

$$n \times 3 \pmod{11} = 9$$

or equivalently

$$9 = n \times 3 - 11 \times k$$

By inspection we can see that  $n = 3$  doesn't work,  $n = 4$  doesn't work but  $n = 5$  works with  $k = 2$  since  $25 \pmod{11} = 25 - 22 = 3$

Performing Euclid's algorithm to find  $\gcd(27,11)$  we have

$$\begin{aligned} 27 &= 11 \times 2 + 5 & 5 &= 27 - 2 \times 11 \\ 11 &= 5 \times 2 + 1 & 1 &= 11 - 2 \times 5 \end{aligned}$$

We want to write 3 in terms of multiples of 5 and 11.