ISC 5935 - Computational Tools for Finite Elements
Homework #3
Assigned 17 September 2014, Due 24 September 2014
http://people.sc.fsu.edu/~jburkardt/classes/fem_2014/homework3.pdf

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1. As written, the finite element program that I gave you prints out something it calls "the error", which is simply the difference between the finite element solution $u^h(x)$ and the exact solution $u(x)$ at each node. This is easy to compute, because at the i-th node, the finite element solution is simply equal to the i-th finite element coefficient.

Mathematically, there are more meaningful error measures: The L2 norm of the error is

$$L2(u - u^h) = ||u - u^h||_{L2} = \sqrt{\int_a^b (u(x) - u^h(x))^2 \quad dx}$$

The H1 seminorm of the error is

$$H1(u - u^h) = ||u - u^h||_{H1} = L2(u_x - u_x^h) = \sqrt{\int_a^b (u_x(x) - u_x^h(x))^2 \quad dx}$$

Luckily, we already know how to estimate integrals; to evaluate the H1 norm, we need to know not only a formula for the exact solution $u(x)$ but also for its derivative $u_x(x)$.

Modify the original finite element program so that it computes and prints the L2 norm and H1 seminorm of the error. Note that, instead of a table of errors at nodes, your error result will simply be two numbers. Run the program for 6, 11, and 21 nodes. **Turn in** a table of the error norms for these 3 cases. Keep a copy of your program in case there are questions!

As a hint, here are the results I got, to three decimal places:

Nodes	L2	H1
6	0.017	0.275
11	0.004	0.138
21	0.001	0.069

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2. Make a modified copy of your finite element program, perhaps called **case2.py**. Change the code to solve the following two point boundary value problem:
Find a function u defined on [0,1] that is twice-continuously differentiable, such that:

$$\begin{aligned} -u'' - 36u &= -128\sin 10x - 448\sin 22x, 0 < x < 1, \\ u(0) &= 0, \\ u(1) &= 0.817478 \end{aligned}$$

Changes you must make include the following:

- In the loop on **j_local**, you need to add a formula to compute phij = $\phi_j(x)$.
- The system matrix is now more complicated. Each increment now will involve **phiip * phijp - 36 * phii * phij**. Make sure the quadrature weight is applied to the total increment, not just the first term!
- The boundary conditions are no longer set using the **exact_fn** function, because we don't have an exact solution.
- We can print the solution, but not the exact solution or error. Modify the print out.
- We can plot the solution (u), but not the exact solution (up). Modify the plot statement.

Run your modified code using 6, 11, 21, 41 nodes. Make a plot of each solution. Do you feel that the 41 node solution is close? **Turn in the 4 plots**. Keep a copy of your program in case there are questions!

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3. Consider the two-point boundary value problem (BVP):
Find a function u defined on [0,1] that is twice-continuously differentiable, such that:

$$\begin{aligned} -u'' + u' + u =& x, 0 < x < 1, \\ u(0) =& 0, \\ u(1) =& 0. \end{aligned}$$

Assume that V^h is an 6-dimensional subspace of V with basis vectors $\psi_1(x), \psi_2(x), ..., \psi_6(x)$ which are the piecewise linear basis functions associated with the mesh $[0, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, 1]$. When solving the discretized weak formulation for the coefficients c, we set up a linear system $A * c = f$. The matrix A can be decomposed as $A = K + L + M$, where the *stiffness matrix* K is:

$$K_{i,j} = \int_0^1 \psi_i'(x)\psi_j'(x)\,dx$$

and the *lucky matrix* L (I made this name up!) is:

$$L_{i,j} = \int_0^1 \psi_i(x)\psi_j'(x)\,dx$$

and the *mass matrix* M is:

$$M_{i,j} = \int_0^1 \psi_i(x)\psi_j(x)\,dx$$

- Using a mesh of 6 nodes, set up the matrices K, L, and M and A, and print these matrices. Do not worry about boundary conditions; in other words, write the first and last equations in the same way as all the others.

- We know that positive definite symmetric matrices have the special property that all their eigenvalues are positive. Use your program to compute and print the 6 eigenvalues of each of the matrices M, L, K and A.

Turn in the values of the eigenvalues for the four matrices. Keep a copy of your program in case there are questions!