A review of finite-element methods for solving the shallow-water equations

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ABSTRACT: The shallow-water equations have been extensively used for a wide variety of coastal phenomena, such as tide-currents, pollutant- dispersion storm-surges, tsunami-wave propagation, etc., to mention but a few. In meteorology the shallow-water equations also known as the primitive barotropic equations have been extensively used to test new numerical solutions for numerical weather prediction models as they posses the same mixture of fast gravity waves and slow Rossby waves as the more complex three dimensional baroclinic primitive equations. In the present survey we will review the application of finite-element methods for solving the shallow-water equations. Various issues such as variable resolution, integral invariant conservation, etc. will be addressed.

1. INTRODUCTION

The solution of the shallow-water equations is of considerable importance for a variety of problems of coastal and environmental engineering such as periodic (tidal) flow, transient wave phenomena (tsunami or shock waves), transient pollutant transport, seiches in ports, etc.

We will review the different finite-element techniques used for coastal phenomena and briefly discuss them in Section 2.

In meteorology use of the finite-element method was initiated by Wang *et al.*, (1972) and the method has since then evolved substantially and is now considered a tool of preference by a sizable number of researchers seeking to solve the two dimensional shallow-water equations. The use of the finite-element method for solving the shallow-water equations in meteorology will be discussed in Section 3.

In Section 4 we will briefly address computational issues related to the use of the finite-element method for solving the shallow-water equations, such as conservation of integral invariants use of different types of elements, variable resolution and the use of finite-element codes on vector supercomputers.

2. SHALLOW-WATER EQUATIONS FOR COASTAL MODELLING

It is commonly known that current flow in estuaries and coastal seas can be described by the shallow-water equations. Assuming vertical density gradients and fluid accelerations are negligible the shallow-water equations can be derived by integrating over the water-depth and assuming hydrostatic pressure.

The equations are

$$\frac{\partial \xi}{\partial t} + \nabla \cdot (H\mathbf{V}) = 0 \quad \text{(conservation of mass)} \tag{1}$$

$$\frac{\partial (H\mathbf{V})}{\partial t} + \nabla \cdot (H\mathbf{V}\mathbf{V}) + \tau H\mathbf{V} + f\mathbf{k} \times H\mathbf{V} + gH\nabla\xi - A = 0 \quad \text{(conservation of momentum)} \tag{2}$$

Where

ξ – is the surface elevation over mean sea level
h – is bathymetry
$\mathbf{H} = h + \tau$ is the total depth of flow
$\mathbf{V}$ – vertically averaged flow velocity
τ – nonlinear bottom friction (Chezy's formula)
f – Coriolis parameter
g – acceleration of gravity
A – atmospheric wind forcing
$\mathbf{k}$ – unit vector in vertical direction

Early finite-element tidal models suffered from severe spurious oscillations. The early applications of these were controlled by large bottom friction coefficients (see Brebbia and Partridge, (1976)) or addition of damping to the model via time-stepping schemes (Kawahara *et al.*, (1978)).

Gray and Lynch (1979) used semi-implicit procedures to partially eliminate spurious modes.

Lynch and Gray (1979) presented a method called the wave equation approach which is in general insensitive to short wavelength oscillations primarily due to good phase speed response.

It is by now accepted that the wave equation approach is capable of noise suppression in finite-element models without the need for artificial or unrealistic damping.

Using different choices of basis functions for elevation and velocity (mixed interpolation) has been suggested by Hood and Taylor (1974) to eliminate $2\Delta x$ oscillations in an attempt to imitate the use of staggered grids in finite-difference approximations.

Kinnmarck and Gray (1984) found that use of a two time-level time-difference approximation, symmetrical but for the nonlinear convective terms can also eliminate spurious $2\Delta t$ oscillations, in the velocity solutions, where the wave equation scheme is used.

Triangular and quadratic Lagrangian isoparametric finite-elements are generally used, (Kinnmarck (1985)). Tidal and storm surge computations using triangular elements and quadratic interpolation were carried out by Dalsecco *et al.*, (1986) with variable resolution. A harbour resonance problem for irregularly shaped harbours using automatic mesh generation was given by Praagman (1986).

A conservative finite-element model of the shallow-water equation with linear triangular elements using a two-step economical algorithm was proposed by Peraire, Zienkiewicz and Morgan (1986), similar to one proposed by Navon and Riphagen (1979) for a compact fourth-order conservative finite-difference approximation of the shallow water equations. The method is easily amenable for vectorization.

Apart from the basic work of Fix (1975), little attention has been paid by coastal and ocean modelers to conservation of integral invariants by finite-element models of the shallow-water equations. A selective lumping finite-element method for shallow-water flow has been extensively tested by Kawahara *et al.*, (1982). Platzman (1981) has considered some response characteristics of finite-element tidal models with the view to partially eliminate small scale errors due to the spatial discretization.

3. SHALLOW WATER EQUATIONS FOR METEOROLOGICAL FLOWS

In meteorology the first application of the finite-element method to the shallow-water equations is due to Wang *et al.*, (1972). They solved the 1–D gravity-wave equations

$$\begin{aligned} u_t + uu_x + ghx &= 0 \\ h_t + uh_x + hu_x &= 0 \end{aligned} \tag{3}$$

where u is the velocity of the fluid in the x-direction, h is the depth of the fluid, and g is the acceleration of gravity. Cubic Hermite functions were used on a uniform grid with a Crank-Nicolson time discretization method. Cullen (1974) used linear equilateral triangles and a leap-frog time scheme to solve the shallow-water equations written in the form:

$$\begin{aligned} u_t + uu_x + vu_y + \phi_x - fv &= 0 \\ v_t + uv_x + vv_y + \phi_y + fu &= 0 \\ \phi_t + (u\phi)_x + (v\phi)_y &= 0 \end{aligned} \tag{4}$$

where u and v are the velocity compounds in the x and y directions, $\phi = gh$ is the geopotential, f is the Coriolis parameter, and h is the depth of the fluid, in a periodic channel on a β-plane where

$$f = f_0 + \beta y \; . \tag{5}$$

Cullen (1974) and Hinsman (1975) used linear equilateral triangles to solve the shallow-water equations on the sphere for the Rossby-Haurwitz waves. Staniforth and Mitchell (1977) used a two dimensional Chapeau basis function to solve the shallow-water equations on a polar stereographic projection. They used vorticity-divergence formulation of the shallow-water equations. A variable resolution integration was performed by Staniforth and Mitchell (1978).

Navon (1979) used an extrapolated Crank-Nicolson scheme with linear triangular elements to solve the shallow-water equations on a β-plane. A selective lumping technique was implemented. Williams (1981) has shown that finite-element formulation of the shallow-water equations using vorticity and divergence as predictive variables on an unstaggered grid does not suffer from the same problems as unstaggered formulations in terms of velocity components (i.e., primitive forms of the equations). He also concluded that if one uses velocity components formulation, one should use them on a staggered grid.

Williams and Zienkiewicz (1981) examined a new formulation of the shallow- water wave equations using different basis functions for the velocity and height fields. They tested a staggered 1–D version of the linearized shallow-water equations.

Based on a proposal by Cullen and Morton (1980), Navon (1983) proposed a Numerov-Galerkin highly accurate finite-element approach to the nonlinear advection operator of the shallow-water equations. A lucid review on the formulation of efficient finite-element codes for flows in regular domains was provided by Staniforth (1987). Neta *et al.*, (1985) studied the linearized shallow-water equations with bilinear rectangular elements for a flow with variable bottom topography.

Neta and Williams (1986) studied various finite-element formulations of the advection equation and found that isoceles triangles and rectangles with bilinear basis functions have better stability and phase speeds. Zienkiewicz and Heinrich (1979) and Zienkiewicz *et al.*, (1986) proposed a finite-element-penalty approach. Steppeler (1987) proposed an energy conserving finite-element scheme for the shallow-water equations.

An in-depth research into the integral invariant conservation properties of different finite-element schemes for the shallow-water equations was conducted by Steppeler, Navon, and Lee (1988). Determination of finite time "blow-up", critical dissipativity required to maintain nonlinear stability and long-term integrations of the finite-element shallow-water equations models were researched.

4. COMPUTATIONAL ASPECTS

While converge and accuracy estimates for Galerkin finite-element methods applied to hyperbolic partial differential equations have been extensively studied (see Dupont (1973) and Thomee and Wendroff (1974)), specific evaluation of different issues for the finite-element methods which solve the shallow-water equations were addressed by Cullen (1976) and Navon (1977).

A Fourier analysis for evaluating the accuracy of finite-element methods for the linearized shallow-water equations, extended to include group velocity was presented by Foreman (1982). His approach was based on similar work by Schoenstadt (1980) and Vichnevetsky and Peiffer (1975). In his research, accuracy was the only consideration in determining a good method–an approach which is good for 1-D considerations. His conclusions point out that the most accurate methods for wave amplitude, phase velocity, and group velocity may not coincide.

He found that for a Galerkin finite-element method with piecewise linear basis functions the most accurate and stable two step method was the Crank-Nicolson method. In a second study Foreman (1984) compared the accuracy and computational cost of three finite-element methods for solving the linearized, two-dimensional shalow-water equations.

He concluded that a finite-element scheme due to Thaker (1978) had some advantage. For triangular elements his analysis indicates that equilateral triangles are the better choice–as they seem to produce isotopic waves when the wave resolution is high.

Weare (1976) considered the computational cost of finite-element and finite difference methods for solving the shallow-water equations. He concluded that finite-element methods are computationally more expensive due to the band algorithms used to solve the sparse matrix equations resulting from this method of discretization. He suggested iterative methods and lumping the mass matrix as possible solutions.

Staniforth (1987) shows that efficient solution algorithms are more easily derived for the rectangular element case than for the triangular element case using a tensor product method. In such a case he shows a way to vectorize the Gaussian elimination procedure. Staniforth (1987) makes also the remark that rectangular elements are inherently more vectorizable because they are well-ordered in memory and easily accessed.

Cullen and Morton (1980) and Navon (1983) applied a two-stage process (compute the derivative, then the product) for calculating the advection term $\frac{u\partial v}{\partial x}$, and have shown that the two-stage method has a smaller coefficient for the asymptotic $O(h^4)$ truncation error .

General conclusions concerning the best solution of the shallow-water equations with Galerkin linear elements point out to the fact that it is necessary either to use velocity components as momentum variables and stagger the nodal points for the free-surface height or to use vorticity and divergence as momentum variables and no staggering (Williams (1981) and Williams and Zienkiewicz (1981)).

As far as stability properties are concerned, finite-element schemes are behaving in a similar way to finite-difference schemes. Generally the same CFL conditions apply, the finite-element schemes having more restrictive coefficients. As they also are the most accurate–there is here a conservation of cost, i.e., the price of increased accuracy is increased cost in terms of more time-steps (Staniforth (1987)).

Conservation of integral invariants of the shallow-water equations related to finite-time "blow-ups", control of non-linear instability due to aliasing of that part of the spectrum generated by the advection product that cannot be resolved by the mesh and the minimal amount of smoothing (filtering) necessary to control this aliasing is treated by Steppeler, Navon and Lu (1988).

As far as time-discretization is concerned, explicit time-differencing schemes are not efficient for stiff sets of equations such as the shallow-water equations.

Navon (1979, 1983) used a Crank-Nicolson method. Semi-implicit methods were used by Staniforth and Daley (1979) for a baroclinic model and by Staniforth and Mitchell (1977) for the finite-element solution of the shallow-water equations. Semi-implicit time-differencing constitutes a good compromise for the shallow-water equations as they require fewer time-steps than explicit methods while their computational cost per time-step is not significantly higher.

As far as variable resolution is concerned, Older (1981) as well as Kelley and Williams (1976) have found that the smoother and slower the change in resolution, the better the forecast; and that one obtains improvement by concentrating high resolution areas in the region of strongest gradients. Also the resolution should be varied with the flow rather than across it. However a smooth transition between fine and coarse resolution is essential.

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