A NUMEROV-GALERKIN TECHNIQUE APPLIED TO A FINITE ELEMENT SHALLOW-WATER EQUATIONS MODEL WITH EXACT CONSERVATION OF INTEGRAL CONSTRAINTS

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INTRODUCTION

In a recent research by Cullen and Morton[1] on finite-element evolutionary error it was suggested that a promising technique for achieving high accu= racy in wave-propagation problems would be to combine the Galerkin product with high order difference approximations to derivatives.

In the present work a high-accuracy two-stage Numerov-Galerkin scheme is used for advective terms in a finite-element shallow-water equations model with a fairly standard test problem in a channel on the rotating earth. An augmented Lagrangian constrained optimization approaches used to enforce an 'a posteriori' conservation of the shallow-water equation integral invariants of mass total energy and enstrophy.

In the first section the Galerkin finite-element model of the shallow-water equations is presented.

In section 2 the two-stage Numerov Galerkin approach is detailed along with the required boundary conditions.

Only the advective terms in the momentum equations are treated by the two-stage Numerov-Galerkin method.

In section 3 an augmented Lagrangian technique using a constrained optimi= zations approach is used for enforcing the conservation of the integral invariants of the shallow-water equations.

Finally, in section 4 some numerical results are presented. More numerical results will be presented at the Conference.

1. THE SHALLOW-WATER FINITE-ELEMENT FORMULATION

The equations describing divergent barotropic motion in an incompressible inviscid fluid with a free-surface are often called the shallow-water equations.

Using a Cartesian coordinate system (see Navon 1979a[2] and Navon and Müller 1979b[3]) the shallow-water equations can be written as follows

$$\begin{aligned} &\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{\partial \phi}{\partial x} - fv = 0 \\ &\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \frac{\partial \phi}{\partial y} + fu = 0 \qquad (1) \\ &\frac{\partial \phi}{\partial t} + \frac{\partial (\phi u)}{\partial x} + \frac{\partial (\phi v)}{\partial y} = 0. \end{aligned}$$

$$0 \le X \le L, \quad 0 \le y \le D, \quad t \ge 0$$

where L and D are the dimensions of a rectangular domain of area $\bar{A} = LD$.

Here u and v are the velocity components in the X and y directions respectively; $\varphi = gh$ is the geopotential h is the depth of the fluid; g is the acceleration of gravity; and f is the Coriolis factor in a rotating frame of reference.

Using linear piecewise polynomials on triangular elements the resulting Galerkin finite-element equations can be written as

$$M(\varphi_j^{n+1} - \varphi_j^n) \quad \frac{-\Delta t}{2} K_1 (\varphi_j^{n+1} + \varphi_j^n) = 0 \tag{2}$$

where M is the mass-matrix

$$M = \iint_A V_j \, V_i \, dA \tag{3}$$

and

$$K_1 = \int \int_A V_j \, V_K \, u_K^* \frac{\partial V_i}{\partial x} dA + \iint V_j \, V_K \, v_K^* \, \frac{\partial V_i}{\partial y} dA \tag{4}$$

where V are the basis functions and

$$u^* = u^{N+\frac{1}{2}} = \tfrac{3}{2}\, u^N - \tfrac{1}{2}\, u^{N-1} + O(\Delta t^2)$$
$$v^* = v^{N+\frac{1}{2}} = \tfrac{3}{2}\, v^N - \tfrac{1}{2} v^{N-1} + O(\Delta t^2) \tag{5}$$

which is a quasi-linearized second-order time-discretization procedure.

The u and v momentum equations are written as

$$M(u_j^{n+1} - u_j^n) + \frac{\Delta t}{2} K_2 (u_j^{n+1} + u_j^n) + \frac{\Delta t}{2} (K_{21}^{n+1} + K_{21}^n) + \Delta t P_2 = 0 \tag{6}$$

$$M(v_j^{n+1} - v_j^n) + \frac{\Delta t}{2} K_3 (v_j^{n+1} + v_j^n) + \frac{\Delta t}{2} (K_{31}^{n+1} + K_{31}^n) + \Delta t P_3 = 0 \tag{7}$$

where the matrices are defined as follows

$$K_2 = \int \int_A u_K^* \, V_K V_i \, \frac{\partial V_j}{\partial x} dA + \int \int_A v_K^* \, V_K V_i \, \frac{\partial V_j}{\partial y} dA \tag{8}$$

$$K_{21}^{n+1} = \int \int_A \varphi_K^{n+1} \, \frac{\partial V_K}{\partial x} V_i \, dA \tag{9}$$

$$P_2 = \int \int_A f v_K^* \, V_K V_i \, dA \tag{10}$$

$$K_3 = \int \int_A u_K^{n+1} \, V_K \frac{\partial V_j}{\partial x} dA + \int \int v_K^* \, V_K \frac{\partial V_j}{\partial x} dA \tag{11}$$

$$K_{31}^{n+1} = \int \int_A \varphi_K^{n+1} \, \frac{\partial V_K}{\partial y} V_i \, dA; \quad P_3 = \int \int_A f u_K^{n+1} \, V_K \, V_i \, dA \tag{12}$$

2. THE TWO-STAGE NUMEROV-GALERKIN SCHEME

The two-stage Galerkin method (see Cullen and Morton[1]) applied to the advec= tive term $u\partial_x v$ is achieved by calculating an intermediary approximation Z to $\partial_x v$ (i.e. the closet piecewise linear approximation) before incorporating it into the Galerkin final approximation to $\frac{u\partial v}{\partial x}$.

As shown by Cullen and Morton[1], if we denote by Z the intermediate approxi= mation to $\frac{\partial v}{\partial x}$ we obtain

$$\frac{1}{6} Z_{j-1} + \frac{2}{3} Z_j + \frac{1}{6} Z_{j+1} = \tfrac{1}{2} h^{-1}(V_{j+1} - V_{j-1}) \qquad (13)$$

The second and final stage is (where $W = u\frac{\partial v}{\partial x}$)

$$\frac{1}{6} W_{j-1} + \frac{2}{3} W_j + \frac{1}{6} W_{j+1} = \frac{1}{12}(U_{j-1}Z_j + U_{j-1}Z_j + U_jZ_{j-1} + U_jZ_{j+1} + U_{j+1}Z_j + U_{j+1}Z_{j+1}) + \tfrac{1}{2}U_jZ_j \qquad (14)$$

Cullen and Morton[1] proved that the truncation error associated with the two-stage Galerkin is almost six-times better than the single-stage Galerkin in the asymptotic limit.

2b. The Two-stage Numerov-Galerkin Scheme for Advective Terms in the Shallow-Water Equations

In this approach we combine the two-stage Galerkin product concept with high order compact implicit (hence the name Numerov) difference approximations to the derivatives. The compact implicit finite difference approximation to the first derivative has a truncation error of $O(h^{4\ell})$ and employs only $2\ell+1$ grid points. (See Schwartz and Wendroff[4]). We found that in order to improve the accuracy of advective terms of the form $u \frac{\partial u}{\partial x}$ in the two-stage Galerkin method it is necessary to use an intermediate approximation to $\frac{\partial u}{\partial x}$ of the order $O(h^8)$ or $\ell=2$ for the Schwartz-Wendroff symbol.

The concise expression of the intermediate compact finite difference approxi= mation to $\frac{\partial u}{\partial x}$ of order $O(h^8)$ is given by

$$\frac{1}{70}\left[\left(\frac{\partial u}{\partial x}\right)_{i+2} + 16\left(\frac{\partial u}{\partial x}\right)_{i+1} + 36\left(\frac{\partial u}{\partial x}\right)_i + 16\left(\frac{\partial u}{\partial x}\right)_{i-1} + \left(\frac{\partial u}{\partial x}\right)_{i-2}\right] = \qquad (15)$$

$$= \frac{1}{84h}[-5u_{i-2} - 32u_{i-1} + 32u_{i+1} + 5u_{i+2}]$$

which necessitates the solution of a pentadiagonal matrix with the entries:

$$\frac{1}{70}\begin{bmatrix} 36 & 16 & 1 & & & 0 \\ 16 & 36 & 16 & 1 & & \\ 1 & 16 & 36 & 16 & 1 & \\ & \ddots & \ddots & \ddots & \ddots & \ddots \\ & & \ddots & \ddots & \ddots & 16 \\ 0 & & 1 & 16 & 36 & \end{bmatrix} \begin{bmatrix} \\ \frac{\partial u}{\partial x} \\ \\ \end{bmatrix} = \frac{1}{84h} \begin{bmatrix} \vdots \\ -5u_{i-2} - 32u_{i-1} \\ +32u_{i+1} + 5u_{i+2} \\ \vdots \end{bmatrix} \qquad (16)$$

The changes in the Galerkin finite-element formulation of the w and v momentum equations are in the advective terms which now take the form:

$$\tilde{K}_2 = [<(u \frac{\partial u}{\partial x})_j^* V_j, V_i > + < (v \frac{\partial u}{\partial y})_j^* V_j, V_i >] \quad (17)$$

$$\tilde{K}_3 = [<(v \frac{\partial v}{\partial y})_j^* V_j, V_i > + <(u \frac{\partial v}{\partial x}) V_j, V_i >] \quad (18)$$

where $< >$ is the inner product or

$$M [(u \frac{\partial u}{\partial x})_j^* + (v \frac{\partial u}{\partial y})_j^*] \quad (19)$$

and

$$M [(v \frac{\partial v}{\partial y})_j^* + (u \frac{\partial v}{\partial x})_j^*] \quad (20)$$

2c. Implementation of the Boundary Conditions

For the particular rectangular channel on a rotating earth periodic boundary conditions are assumed in the x-direction while in the y direction rigid boundary conditions are imposed i.e.

$$v(x,0,t) = v(x,D,t) = 0 \quad (21)$$

This implies that some extraneous boundary conditions are required for the Numerov derivative and Z_1, Z_2, $Z_{N_{y-1}}$ and Z_{N_y} are replaced by $O(h^4)$ one-sided approximations of the derivative i.e. say for $\frac{\partial v}{\partial y}$

$$Z_1 = (-25v_1 + 48v_2 - 36v_3 + 16v_4 - 3v_5)/12h \quad (22)$$

$$Z_2 = (-3v_1 - 10v_2 + 18v_3 - 6v_4 + v_5)/12h \quad (23)$$

$$Z_{N_{y-1}} = (v_{N_{y-4}} + 6v_{N_{y-3}} - 18v_{N_{y-2}} + 10v_{N_{y-1}} + 3v_{N_y})/12h \quad (24)$$

$$Z_{N_y} = (3v_{N_{y-4}} - 16v_{N_{y-3}} - 36v_{N_{y-2}} - 48v_{N_{y-1}} + 25v_{N_y})/12h \quad (25)$$

For the values of v_o and $v_{N_{y+1}}$ one uses a cubic extrapolation i.e.

$$v_{N_{y+1}} = 4v_{N_y} - 6v_{N_{y-1}} + 4v_{N_{y-2}} - v_{N_{y-3}} \quad (26)$$

Similar formulas are used for $\frac{\partial u}{\partial y}$.

For the solution of the pentadiagonal system a generalization of the Thomas algorithm is used and for the cyclic boundaries the resulting cyclic pentadiagonal matrix is solved by a generalization of the Ahlberg-Nielson-Walsh[5] algorithm.

3. AN AUGMENTED LAGRANGIAN PENALTY METHOD FOR ENFORCING DISCRETE CONSERVATION OF INTEGRAL INVARIANTS

The shallow-water equations have three main integral invariants namely total mass

$$H = \int_0^L \int_0^D \frac{h\,dxdy}{A} \tag{27}$$

total energy

$$E = \tfrac{1}{2} \int_0^L \int_0^D (u^2+v^2+\phi)\, \frac{\phi}{g}\, dxdy \qquad \phi = gh \tag{28}$$

and potential enstrophy

$$Z = \tfrac{1}{2} \int_0^L \int_0^D (\frac{Q^2}{h}) dxdy = \tfrac{1}{2} \int_0^L \int_0^D (\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} + f)^2\, h^{-1}\, dxdy \tag{29}$$

Some 'a posteriori' methods were described by Navon (1981)[6]. Here we propose an augmented Lagrangian multiplier and penalty method. Our augmented Lagrangian takes the form

$$L_r(x) = f(x) + \frac{1}{2r}\, |e(x)|^2 \tag{30}$$

or

$$L_r(x,u) = f(x) + u^T\, e(x) + \frac{1}{2r} |e(x)|^2 \tag{30a}$$

where

$$f = \sum_{i=1}^{Nx} \sum_{j=1}^{Ny} [\alpha(u-\tilde{u})^2 + \alpha(v-\tilde{v})^2 + \beta(h-\tilde{h})^2]_{ij} \tag{31}$$

α and β are weights, $(\tilde{u},\tilde{v},\tilde{h})_{ij}$ are the predicted variables at the n-th time-step of the integration of the shallow-water equations while $(u,v,h)_{ij}$ are the adjusted values by the constrained optimization method enforcing the conservation of integral invariants. We consider the problem

$$\begin{aligned} &\text{minimize } f(\underline{x}) \\ &\text{subject to } e(\underline{x}) = 0 \end{aligned} \tag{32}$$

where

$$\underline{x} = (u_{11} \ldots u_{N_x,N_y}, v_{11} \ldots v_{N_x,N_y}, h_{11} \ldots h_{N_x N_y}) \tag{33}$$

and $e(\underline{x})$ are the nonlinear equality constraints given by

$$e(\underline{x}) = \begin{cases} E^n - E^o = 0 \\ Z^n - Z^o = 0 \\ H^n - H^o = 0 \end{cases} \tag{34}$$

where the superscripts n and o stand for the time $u\Delta t$ and the initial time respectively.

In eqn (30a) r is a penalty parameter and $\underline{u}$ a Lagrange multiplier vector.

For updating the multipliers and the penalty parameters we follow the

Bertsekas[7], method and use the following updating formulas

$$u_{k+1} = u_K + r_K^{-1}\, e(x_K) \tag{35}$$

for the Lagrange multipliers

and

$$r_{k+1} = \begin{cases} \beta r_K & \text{if } |e(x_K,u_K)| > \gamma |e(x_{K-1},u_{K-1})| \\ r_K & \text{if } |e(x_K,u_K)| \le \gamma |e(x_{K-1},u_{K-1})| \end{cases} \tag{36}$$

with $\beta = 0.1$ and $\gamma = 0.25$.

NUMERICAL RESULTS

The Test Problem

We solved the shallow-water equations in a channel of width D = 4400 km and periodic in the x-direction the initial conditions are derived from a height field condition N = 1 of Grammeltvedt[8] given by

$$h(x,y) = H_o + H_1 \tanh\left(\frac{9(D/2-y)}{2D}\right) + H_2\, \mathrm{sech}^2\left(\frac{9(D/2-y)}{D}\right) \cdot \sin\left(\frac{2\pi x}{L}\right) \tag{37}$$

The initial velocity fields were derived from the initial height field using the geostrophic relationship

$$u = \left(\frac{-g}{f}\right)\frac{\partial h}{\partial y} \qquad v = \left(\frac{g}{f}\right)\frac{\partial h}{\partial x} \tag{38}$$

while the parameters here are:

$H_o = 2000$ m $\quad H_2 = 133$ m $\quad f = 10^{-4}\,\mathrm{sec}^{-1}$

$H_1 = 220$ m $\quad g = 10\ \mathrm{m\ sec}^{-2}$

We used a grid space of 400 km and the time-step was 1800 sec.

We compared our Numerov Galerkin two-stage technique with a single-stage Galerkin run and with a point multiplication scheme (PMG)(see Cullen and Morton).

For long-term runs the Numerov-Galerkin method remains very stable while the PMG method goes unstable after 5 days.

The Numerov-Galerkin technique turned out to be computationally economic, as it simplified quite a number of element matrices. This resulted in an economy of about 35% of the computational time spent on each time-step.

As far as accuracy is concerned there was only a marginal improvement over the usual single stage Galerkin method. More results on this issue will be presented at the Conference.

All three integral invariants were well conserved for a 10 day integration with the Numerov-Galerkin technique (see Figs.1-3) and an increase in accu= racy with the imposition of exact conservation via constrained optimization was obtained.

More specifically, when enforced conservation of enstrophy and mass was applied, combined with the application of a Shuman filter every 6 time steps, an improvement of 50% in accuracy (as defined in Navon[6]) was obtained by using the Numerov-Galerkin method as compared with the single-stage Galerkin after 2 days. This improvement increased to 100% after 4 days of numerical integration.

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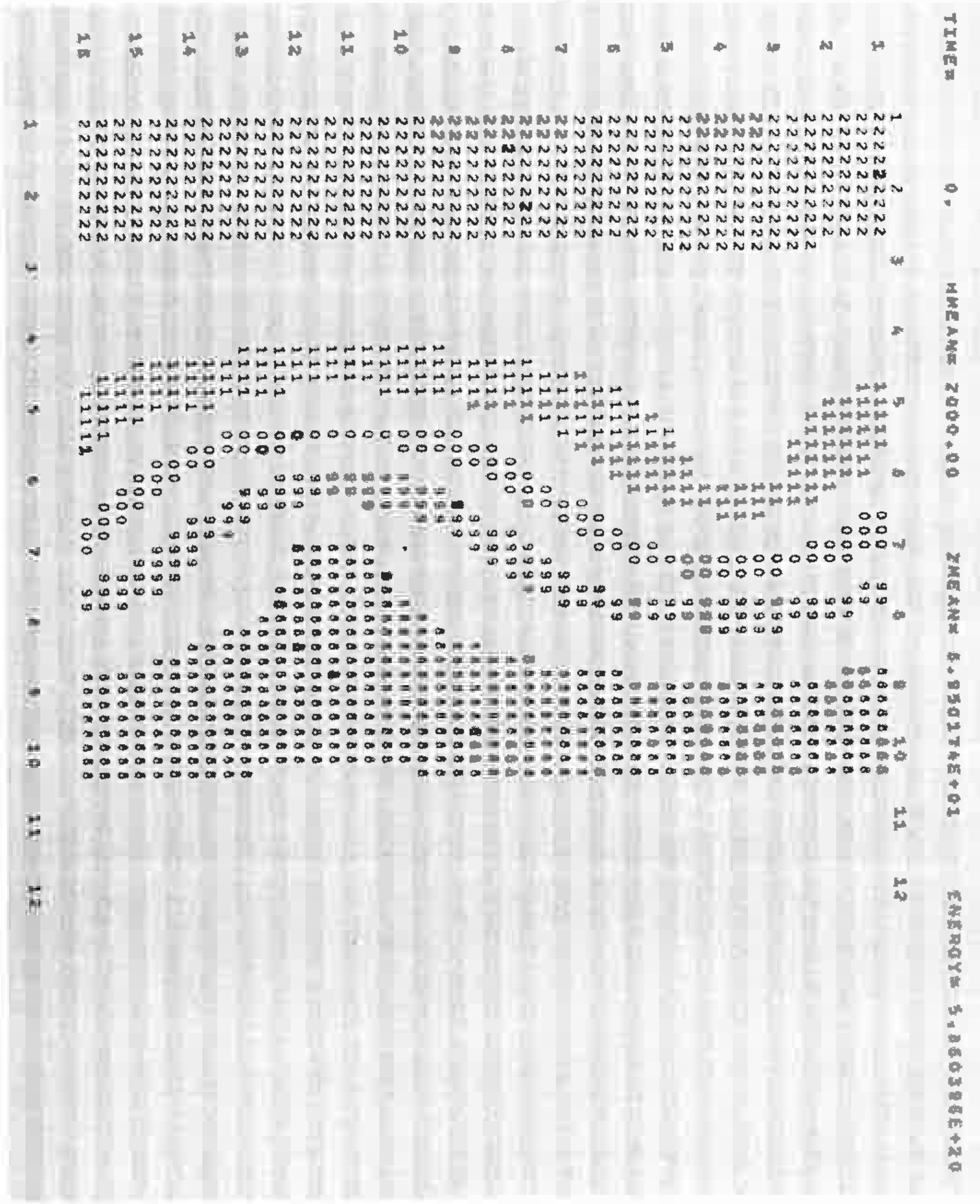

Fig. 1: The initial distribution of the height field depicted by isopleths drawn at 50 m intervals. The initial values of mass (h mean) enstrophy (z mean) and total energy (energy) invariants are also displayed.

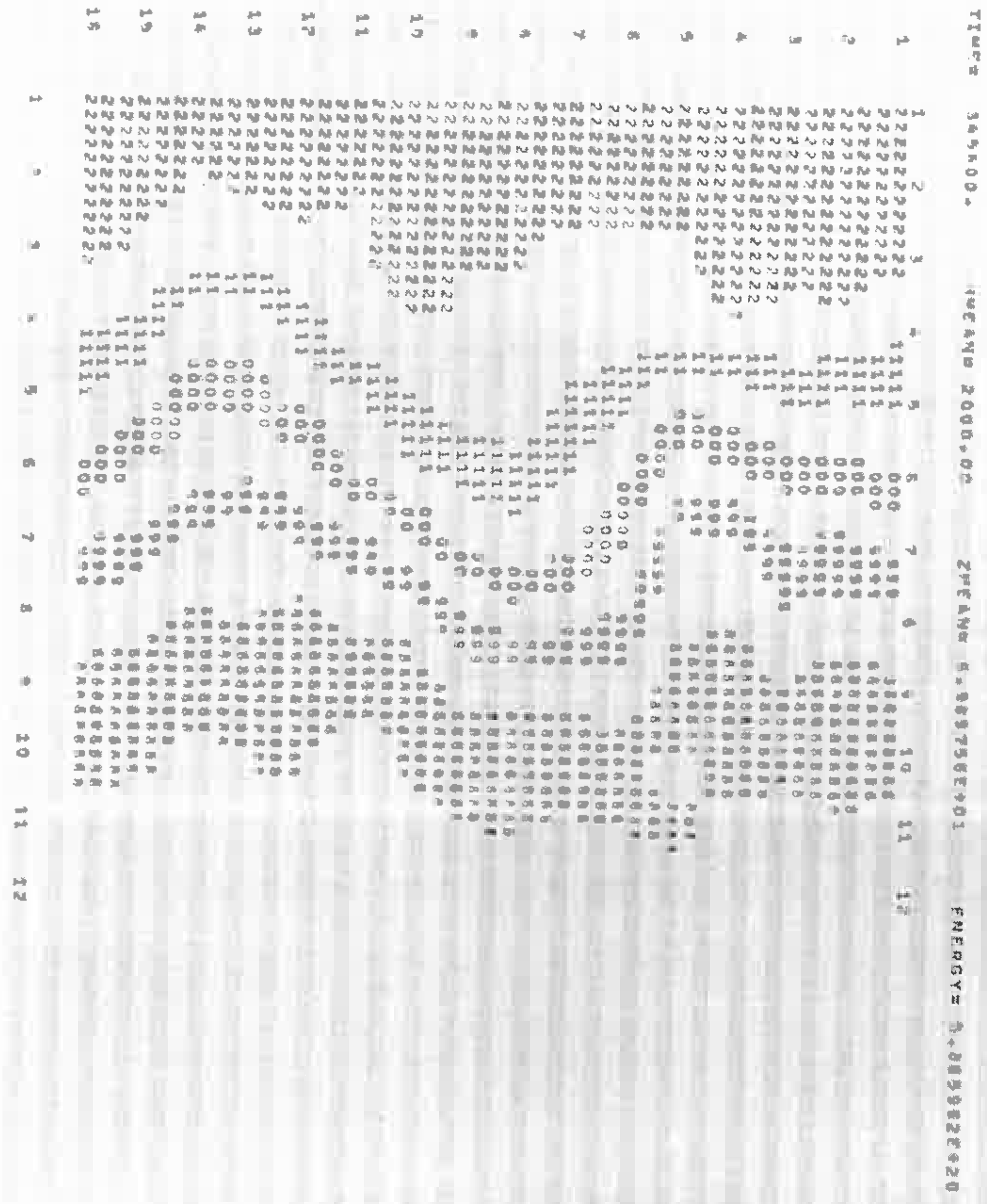

Fig.2: A 4-day forecast of the height field using the two-stage Numerov-Galerkin method.

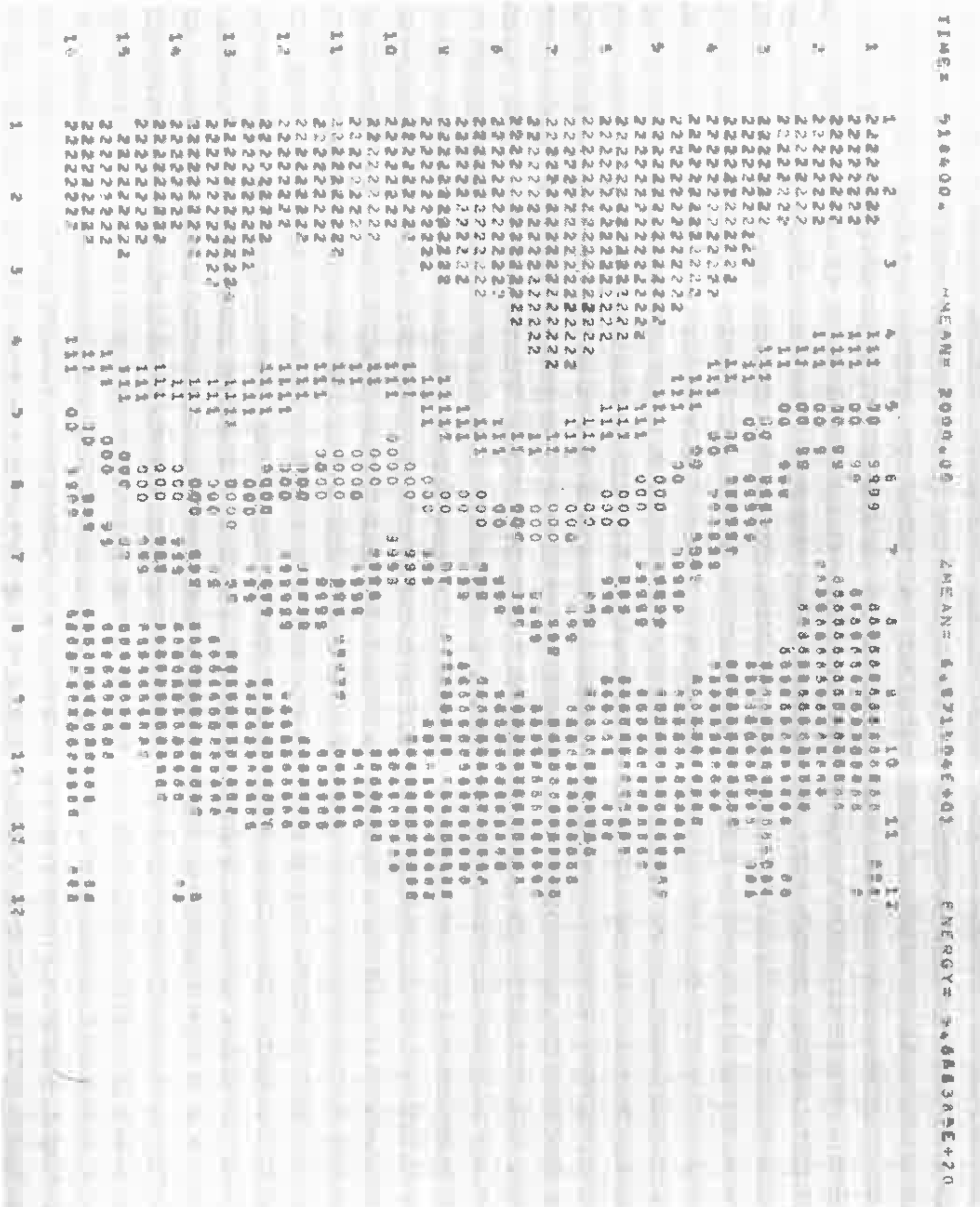

Fig.3: A 6-day forecast of the height field using the two-stage Numerov-Galerkin method.