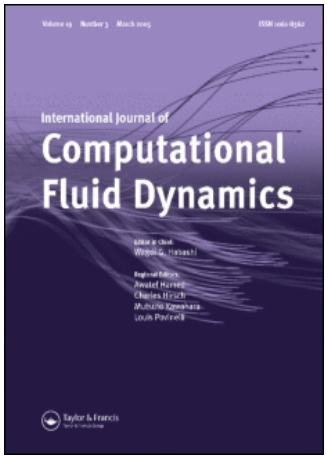

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Estimation of goal functional error arising from iterative solution of Euler equations

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Estimation of the error arising in the cost (goal) functional due to stopping the iterative process is considered for a steady problem solved by temporal relaxation. The functional error is calculated using an iteration residual along with related adjoint parameters. Numerical tests demonstrate the applicability of this approach for the steady 2D Euler equations.

Keywords: iteration error; goal functional; adjoint equations; Euler equations

1. Introduction

The quantitative evaluation of errors caused by different components of a numerical algorithm including the error of approximation and iteration error is of significant current interest (AIAA Standard 1998, Oberkampf and Blottner 1998, Roache 1998). Iterative methods are commonly used for solving steady CFD problems. One of the simplest techniques involves the temporal evolution from an initial guess to obtain a steady solution. This approach implies performing iterations along the time coordinate or a certain pseudo-temporal variable (Roache 1976, Samarskii 2001). Different variants of preconditioning (Pulliam and Chaussee 1981, Allmaras 1993, Turkel 1993, Jameson and Caughey 2001, Turkel and Vatsa 2005) are used to improve the relaxation rate. Commonly used *a priori* estimates of iteration convergence (Samarskii 2001) (linking the error with iteration residual in certain norms) contain constants that are unknown in the general case (for nonlinear non self-adjoint operators). Very often, the iterations are terminated when some semi-empirical convergence criterion (for example, $\max_i|\rho_i^{n+1} - \rho_i^n| \leq 10^m$ (Roache 1976)) is satisfied. This entails a small iteration error; however the exact magnitude of this error remains unknown. In the present paper, we estimate the error of a cost (valuable from a practical viewpoint) functional in the form of *a posteriori* error estimation obtained via adjoint parameters (Wigner 1945, Marchuk 1995, Giles and Pierce 1997, Giles and Pierce 1999, Alekseev 2000, Fursikov 2000, Oden and Vemaganti 2000, Becker and Rannacher 2001, Giles and Suli 2002, Oden and Prudhomme 2002, Zuazua 2002, Braack and Ern 2003, Cnossen *et al.* 2003, Gunzburger 2003, Hartmann and Houston 2003, Giles *et al.* 2004, Pierce and Giles 2004, Alekseev and Navon 2005a, 2005b, Alekseev 2006) and the residual of the iterations. From another viewpoint, the approach used here may be considered as an estimation of the error caused by the variation of the physical model. A sizeable number of publications cover the impact of physical model variation (Oden and Vemaganti 2000, Giles and Suli 2002, Oden and Prudhomme 2002, Braack and Ern 2003, Cnossen *et al.* 2003, Giles *et al.* 2004, Alekseev and Navon 2005a, Alekseev 2006) using the adjoint equations. The results provided by 'coarse' and 'fine' physical models are compared by Oden and Vemaganti (2000) and Oden and Prudhomme (2002) for several problems including the flow of a viscous incompressible fluid governed by the Navier–Stokes or Stokes equations models. The influence of a coefficient's oscillations and nonlinearity for the Poisson and convection–diffusion–reaction equations is estimated in (Braack and Ern 2003), while the deviation of solutions governed by Helmholtz and Poisson equations models is considered in (Cnossen *et al.* 2003). Alekseev and Navon (2005a) discuss the impact of viscosity on the flow parameters by comparing the Euler and parabolised Navier–Stokes equations. In the present paper, we address the issue of comparing steady and unsteady inviscid gas flows. The flow density at a certain reference point is chosen here as the cost functional. This choice is not central to the method and is used due to its convenience for numerical tests. This approach was also employed for the heat transfer equation and pointwise temperature in (Alekseev 2006).

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2. Algorithm outline

Let us consider briefly the formal scheme of the adjoint *a posteriori* error estimation in accordance with work by Giles and Pierce (1997), Giles and Pierce (1999), Oden and Vemaganti (2000), Becker and Rannacher (2001), Giles and Suli (2002), Oden and Prudhomme (2002), Braack and Ern (2003), Hartmann and Houston (2003), Giles *et al.* (2004) and Pierce and Giles (2004). This approach is adopted for the present paper purposes and assumes the same form of accounting for the impact of both unsteady terms and truncation error.

Let the problem of interest (direct one) be governed by the equation

$$N(f) = w \quad \text{in} \quad \Omega \subset R^n, \tag{1}$$

with boundary conditions

$$B(f) = e \quad \text{on} \quad \partial\Omega, \tag{2}$$

where N is a nonlinear differential operator $(H^k(\Omega) \rightarrow L_2(\Omega))$, f denotes a set of physical parameters $(f \in H^k(\Omega))$, w, e are the control parameters $(w \in L_2(\Omega), e \in L_2(\partial\Omega))$. The operators $N(f)$, $B(f)$ are considered to be Frechet differentiable, while their corresponding derivatives are denoted as $N_f(f)$, $B_f(f)$, respectively. For the sake of brevity, we will also use for these operators the following notations: Nf, Bf and N_f, B_f, respectively.

Consider also a Frechet differentiable cost functional $\varepsilon(\cdot): L_2(\Omega) \rightarrow R^1$.

Apart from the value of this functional, its sensitivity to the control parameter variation is also of practical significance. We introduce the adjoint parameters $\Psi \in H^1(\Omega)$ and formulate the Lagrangian

$$L = \varepsilon + (Nf - w, \Psi)_{L_2(\Omega)} + (Bf - e, \Psi)_{L_2(\partial\Omega)}. \tag{3}$$

We want to track the impact of the control parameter variations Δw, Δe on the cost functional.

The variation of the Lagrangian assumes the form

$$\begin{aligned}\Delta L =& (\varepsilon_f, \Delta f)_{L_2(\Omega)} + (N_f \Delta f - \Delta w, \Psi)_{L_2(\Omega)} \\ &+ (Nf - w, \Delta\Psi)_{L_2(\Omega)} + (Bf - e, \Delta\Psi)_{L_2(\partial\Omega)} \\ &+ (B_f \Delta f - \Delta e, \Psi)_{L_2(\partial\Omega)}.\end{aligned} \tag{4}$$

It may be seen that the variation of the cost functional is equal to the variation of the Lagrangian on the solutions of the direct problem and an additional problem, describing the perturbations:

$$N_f(f)\Delta f = \Delta w \quad \text{in} \quad \Omega, \tag{5}$$

$$B_f(f)\Delta f = \Delta e \quad \text{on} \quad \partial\Omega. \tag{6}$$

It is convenient to use Gateaux derivatives (derivatives along the direction of disturbance $\Delta_f = \Delta f / \|\Delta f\|$) of the functional (and Lagrangian) instead of variations and equations for the perturbations

$$\Delta_\varepsilon = \varepsilon_f(f)\Delta_f = \lim_{s \rightarrow 0} \frac{\varepsilon(f + s\Delta_f) - \varepsilon(f)}{s}, \tag{7}$$

and a Gateaux derivative of the operator

$$N_f \Delta_f = \lim_{s \rightarrow 0} \frac{N(f + s\Delta_f) - N(f)}{s}. \tag{8}$$

The Gateaux derivative of the direct problem operators is denoted as the *tangent linear problem* assuming the following form:

$$N_f(f)\Delta_f = \Delta_w \quad \text{in} \quad \Omega, \quad \|\Delta_w\| = 1, \tag{9}$$

with boundary conditions

$$B_f(f)\Delta_f = \Delta_e \quad \text{on} \quad \partial\Omega, \quad \|\Delta_e\| = 1. \tag{10}$$

This statement is more useful compared with (5)–(6) since it does not involve assumptions of smallness of the disturbances.

The Gateaux derivative of the cost functional is a linear continuous functional that may be formulated as a Riesz-representation using an inner product in $L_2(\Omega)$

$$\Delta_\varepsilon = (\varepsilon_f, \Delta_f)_{L_2(\Omega)}. \tag{11}$$

Correspondingly, the Gateaux derivative of the Lagrangian assumes the form

$$\begin{aligned}\Delta_L = &(\varepsilon_f, \Delta_f)_{L_2(\Omega)} + (N_f \Delta_f - \Delta_w, \Psi)_{L_2(\Omega)} \\ &+ (B_f \Delta_f - \Delta_e, \Psi)_{L_2(\partial\Omega)}.\end{aligned} \tag{12}$$

Using the bilinear identity $(N_f \Delta_f, \Psi)_{L_2(\Omega)} = (N_f^* \Psi, \Delta_f)_{L_2(\Omega)} + (\Gamma \Delta_f, \Psi)_{\partial\Omega}$ implemented by integration by parts, we may recast (12) as

$$\begin{aligned}\Delta_L &= (\varepsilon_f, \Delta_f)_{L_2(\Omega)} + (N_f \Delta_f, \Psi)_{L_2(\Omega)} \\ &\quad - (\Delta_w, \Psi)_{L_2(\Omega)} + (B_f \Delta_f, \Psi)_{L_2(\partial\Omega)} \\ &\quad - (\Delta_e, \Psi)_{L_2(\partial\Omega)} \\ &= (\varepsilon_f, \Delta_f)_{L_2(\Omega)} + (\Delta_f, N_f^* \Psi)_{L_2(\Omega)} \\ &\quad - (\Delta_w, \Psi)_{L_2(\Omega)} + (\Gamma \Delta_f, \Psi)_{L_2(\partial\Omega)} \\ &\quad + (\Delta_f, B_T^* \Psi)_{L_2(\partial\Omega)} - (\Delta_e, \Psi)_{L_2(\partial\Omega)} \\ &= \left(\Delta_f, N_f^* \Psi + \varepsilon_f\right)_{L_2(\Omega)} - (\Delta_w, \Psi)_{L_2(\Omega)} \\ &\quad + \left(\Delta_f, B_f^* \Psi + \Gamma^* \Psi\right)_{L_2(\partial\Omega)} - (\Delta_e, \Psi)_{L_2(\partial\Omega)}.\end{aligned} \tag{13}$$

The derivative of the Lagrangian along the control parameter disturbance may be expressed as

$$\Delta_L = -(\Delta_w, \Psi)_{L_2(\Omega)} - (\Delta_e, \Psi)_{L_2(\partial\Omega)}. \tag{14}$$

It is valid under the condition (that forms the adjoint equations)

$$\begin{aligned} &N_f^* \Psi + \varepsilon_f = 0 \quad \text{in} \quad \Omega, \\ &B_f^* \Psi + \Gamma^* \Psi = 0 \quad \text{on} \quad \partial\Omega. \end{aligned} \tag{15}$$

The variation of the Lagrangian subject to the perturbations Δw, Δe equals

$$\begin{aligned} \Delta\varepsilon = \Delta L &= -\alpha(\Delta_w, \Psi)_{L_2(\Omega)} - \beta(\Delta_e, \Psi)_{L_2(\partial\Omega)} \\ &= -(\Delta w, \Psi)_{L_2(\Omega)} - (\Delta e, \Psi)_{L_2(\partial\Omega)}, \end{aligned} \tag{16}$$

where $\alpha = \|\Delta w\|$, $\Delta w = \alpha \Delta_w$.

The present paper aims to estimate the impact of an unsteady term on the cost functional. In the numerical tests to be presented, we compare the above impact with the impact of the truncation error of the finite-difference scheme. In both problems the control parameters should have a source-like form. Thus, we will only consider the impact of sources Δw and reduce (16) to the form $\Delta\varepsilon = -(\Delta w, \Psi)_{L_2(\Omega)}$.

2.1 Impact of the truncation error

It is natural to compare the numerical effect of the iteration termination to other numerical errors. For these reasons, the influence of the truncation error on the cost functional is accounted for according to approaches presented by Giles and Pierce (1997), Giles and Pierce (1999), Giles and Suli (2002), Venditti and Darmofal (2002), Hartmann and Houston (2003), Giles *et al.* (2004), Pierce and Giles (2004), Alekseev and Navon (2005a, 2005b) and Alekseev and Navon (2006). We consider that the numerical solution of the direct problem is assuming the following form

$$\begin{aligned} &Nf = w \quad \text{in} \quad \Omega \subset R^n, \\ &f(\partial\Omega) = f_B(x) \in L_2(\partial\Omega). \end{aligned} \tag{17}$$

The numerical solution is provided by the finite-difference equation

$$N_h f_h = w. \tag{18}$$

As a result of its solution we obtain a grid function f_h. We assume the existence of a smooth enough function $f \in H^{k+n}(\Omega)$ that coincides with the grid function at the discretisation nodes. Finite differences in $N_h f_h$ may be expanded using the Taylor series. This provides us with a differential approximation of the finite-difference scheme

$$Nf + \delta_h(f) = w, \quad \text{or} \quad Nf = w + \Delta w. \tag{19}$$

Here, $\delta_h(f)$ is the approximation error containing the leading terms of the Taylor expansion and it serves as the disturbing term $\Delta w = -\delta_h(f)$. The corresponding disturbances are governed by the expression

$$\begin{aligned} &N_f \Delta f = \Delta w = -\delta_h(f), \\ &\Omega \subset R^n, \quad \Delta f(\partial\Omega) = 0. \end{aligned} \tag{20}$$

According to (16) the variation of the cost functional caused by the approximation error may be expressed by

$$\Delta\varepsilon = -(\Delta w, \Psi)_{L_2(\Omega)} = \int_\Omega \delta_h(f) \Psi \, d\Omega. \tag{21}$$

The adjoint equations are modified for this problem as follows:

$$N_f^* \Psi + \varepsilon_f = 0, \quad \text{in} \quad \Omega, \quad \Psi = 0, \quad \text{on} \quad \partial\Omega. \tag{22}$$

2.2 Impact of the unsteady terms

Let us consider a brief formal scheme of the above-mentioned approach for temporal relaxation. We solve a steady nonlinear differential problem (coinciding with (17))

$$\begin{aligned} &N\tilde{f} = w \quad \text{in} \quad \Omega \subset R^n, \\ &\tilde{f}(\partial\Omega) = \tilde{f}_B(x) \in L_2(\partial\Omega). \end{aligned} \tag{23}$$

For solving this problem, time iterations are used that are equivalent to the following unsteady statement

$$\begin{aligned} &\partial f/\partial t + Nf = w \quad \text{in} \quad Q = \Omega \times (0, t_f), \\ &f(\partial\Omega) = f_B(x) \in L_2(\partial\Omega); \\ &\text{with an initial guess} \\ &f(\Omega, 0) = f_0(x) \in L_2(\Omega). \end{aligned} \tag{24}$$

Consider $N\tilde{f} = w$ as an exact equation and $Nf = w - \partial f/\partial t$ as the perturbed one. The exact and perturbed solutions are related by $f(t,x) = \tilde{f}(x) + \Delta f(t,x)$. Then, for a fixed time t the disturbance $\Delta f(t,x)$ may be obtained as the solution of the steady equation

$$\Delta w - N_f \Delta f = 0 \quad \text{in} \quad \Omega \subset R^n, \quad \Delta f(\partial\Omega) = 0, \tag{25}$$

where $\Delta w(t,x) = \partial f/\partial t$ is considered as a source-like disturbance.

So, for a fixed moment t, we obtain practically the same problem as (20) or (5), (6) with the only difference being in form of the sources. The variation of a cost functional may be represented by using the inner product in $L_2(\Omega)$ as

$$\Delta\varepsilon = (\Delta f, \varepsilon_f)_{L_2(\Omega)} = (\Delta w, \Psi)_{L_2(\Omega)}, \tag{26}$$

where Ψ is the solution of adjoint problem

$$N_f^* \Psi - \varepsilon_f = 0 \quad \text{in} \quad \Omega \subset R^n, \quad \Psi(\partial\Omega) = 0. \tag{27}$$

Thus, the cost functional variation caused by the iteration residual may be expressed as

$$\Delta\varepsilon = \int_\Omega \Delta w(t,x)\Psi \, \mathrm{d}\Omega. \tag{28}$$

The adjoint problem can be solved using another iterative method; herein the time relaxation of the following form was used:

$$\begin{aligned} &\partial\Psi/\partial\tau + N_f^*\Psi + \varepsilon_f = 0 \quad \text{in} \quad Q = \Omega \times (0, \tau_f), \\ &\Psi(\partial\Omega) = 0, \quad \Psi(\Omega, 0) = 0. \end{aligned} \tag{29}$$

It should be noted that problem (29) is not related to problem (24) neither by a common temporal interval nor by the form of iterations.

3. Test problem

Let us consider the approach described above for a steady 2D compressible inviscid flow. The iterations are based on temporal relaxation using the unsteady form of the 2D Euler equations

$$\frac{\partial\rho}{\partial t} + \frac{\partial(\rho U^k)}{\partial X^k} = 0; \tag{30}$$

$$\frac{\partial(\rho U^i)}{\partial t} + \frac{\partial(\rho U^k U^i + P\delta_{ik})}{\partial X^k} = 0; \tag{31}$$

$$\frac{\partial\left(\rho(e + U^i U^i/2)\right)}{\partial t} + \frac{\partial(\rho U^k(\gamma e + U^i U^i/2)\big)}{\partial X^k} = 0; \tag{32}$$

$$\begin{aligned} &P = (\gamma - 1)\rho e, \\ &(x, y) \in \Omega = (0 < x < X_{\max}; 0 < y < Y_{\max}), \\ &(0 < t < t_f). \end{aligned}$$

Here, ρ-density, X^k-coordinates $(X^1 = x, X^2 = y)$, U^k-velocity components, $\theta = 1/2(U^i U^i)$, P-pressure, $\gamma = C_p/C_v$, $e = C_v T$-inner energy, $h(\rho, P) = \gamma e$-enthalpy, and $h_0 = \theta + h$-total enthalpy, where summation over repeating indices is assumed.

The boundary conditions for inflow and lateral sides $(x = 0; y = 0; y = Y_{\max})$ were considered as steady ones corresponding to inflow parameters, while for the outflow boundary $(x = X_{\max})$ the condition $\partial f/\partial x = 0$ was imposed.

The calculation of the steady flow-field was performed using time evolution starting from a spatially uniform initial guess.

The pointwise density was used as the goal functional $\rho(x^{est}, y^{est})$

$$\varepsilon = \rho(x^{est}, x^{est}) = \int_\Omega \rho(x,y)\delta(y - y^{est})\delta(y - y^{est})\mathrm{d}x\,\mathrm{d}y. \tag{33}$$

The present form of the cost functional is selected by considering the simplicity of comparisons with exact (analytical) values. It is a rather difficult one from a computational viewpoint due to the singularity. A large set of practically important functionals is formed by some integration (usually of the pressure) and does not engender singularities in the adjoint statements.

The corresponding adjoint problem may be obtained using the standard approach (Marchuk 1995, Giles and Pierce 1997, Giles and Pierce 1999, Alekseev 2000, Fursikov 2000, Oden and Vemaganti 2000, Becker and Rannacher 2001, Giles and Suli 2002, Oden and Prudhomme 2002, Zuazua 2002, Braack and Ern 2003, Cnossen *et al.* 2003, Gunzburger 2003, Hartmann and Houston 2003, Giles *et al.* 2004, Pierce and Giles 2004, Alekseev and Navon 2005a, 2005b, Alekseev 2006) and assumes the form

$$\begin{aligned} &\frac{\partial\Psi_\rho}{\partial\tau} + (-U^2 + \theta(\gamma - 1))\frac{\partial\Psi_U}{\partial x} - UV\frac{\partial\Psi_V}{\partial x} \\ &\quad + (-Uh_0 + U\theta(\gamma - 1))\frac{\partial\Psi_E}{\partial x} + (-UV)\frac{\partial\Psi_U}{\partial y} \\ &\quad + (-V^2 + \theta(\gamma - 1))\frac{\partial\Psi_V}{\partial y} + (-Vh_0 + V\theta(\gamma - 1)) \\ &\quad \times\frac{\partial\Psi_E}{\partial y} - \delta(X - X^{est})\delta(Y - Y^{est}) = 0, \end{aligned} \tag{34}$$

$$\begin{aligned} &\frac{\partial\Psi_U}{\partial\tau} + \frac{\partial\Psi_\rho}{\partial x} + (2U - U(\gamma - 1))\frac{\partial\Psi_U}{\partial x} \\ &\quad + V\frac{\partial\Psi_V}{\partial x} + (h_0 - U^2(\gamma - 1))\frac{\partial\Psi_E}{\partial x} + V\frac{\partial\Psi_U}{\partial y} \\ &\quad - U(\gamma - 1)\frac{\partial\Psi_V}{\partial y} - UV(\gamma - 1)\frac{\partial\Psi_E}{\partial y} = 0, \end{aligned} \tag{35}$$

$$\begin{aligned} &\frac{\partial\Psi_V}{\partial\tau} - V(\gamma - 1))\frac{\partial\Psi_U}{\partial x} + U\frac{\partial\Psi_V}{\partial x} - UV(\gamma - 1))\frac{\partial\Psi_E}{\partial x} \\ &\quad + \frac{\partial\Psi_\rho}{\partial y} + U\frac{\partial\Psi_U}{\partial y} + (2V - V(\gamma - 1))\frac{\partial\Psi_V}{\partial y} \\ &\quad + (h_0 - V^2(\gamma - 1))\frac{\partial\Psi_E}{\partial y} = 0, \end{aligned} \tag{36}$$

$$\frac{\partial \Psi_E}{\partial \tau} + (\gamma - 1)\frac{\partial \Psi_U}{\partial x} + \gamma U \frac{\partial \Psi_E}{\partial x} + (\gamma - 1)\frac{\partial \Psi_V}{\partial y} + \gamma V \frac{\partial \Psi_E}{\partial y} = 0, \tag{37}$$

$$\text{with initial conditions :} \quad \Psi_{\rho,U,V,E}|_{\tau=0} = 0; \quad \text{and} \tag{38}$$

$$\text{boundary conditions } (y = 0; y = Y_{\max}) : \quad \Psi_{\rho,U,V,E}|_{\partial\Omega} = 0. \tag{39}$$

The parameters $(\Psi_\rho, \Psi_U, \Psi_V, \Psi_E)$ are the adjoint analogues of density, velocity components and energy, respectively. The coefficients in these equations are composed of gas dynamics parameters obtained from the solution of (30)–(32) at the estimated moment. The problem (34)–(39) is linear, so there is no shock wave formation within the computational field. The discontinuities may be provided only by the boundary conditions (Kraiko 1979) or the cost functional. The Dirac δ-form source in the equation for Ψ_ρ corresponds to the location of the reference point. A mollification of the δ-form source term was used according to Walden (1999) and Tornberg and Engquist (2003) and is based on following form:

$$\delta \sim \exp(-(x - x^{est})^2/\sigma^2 - (y - y^{est})^2/\sigma^2). \tag{40}$$

In numerical tests the value of σ was selected so as to provide a smearing of the source over 2–3 cells ($\sigma \sim h$) thus providing regularity of the numerical solution.

The convergence error estimate (analogue of Equation (28)) has a form:

$$\delta\varepsilon = \int_\Omega \int \left(\Psi_\rho \frac{\partial \rho}{\partial t} + \Psi_U \frac{\partial U}{\partial t} + \Psi_V \frac{\partial V}{\partial t} + \Psi_E \frac{\partial e}{\partial t} \right) \mathrm{d}x\, \mathrm{d}y. \tag{41}$$

This expression determines quantitatively the deviation of the numerical result from an exact steady value due to the termination of iterations (evolution) at moment t.

4. Numerical tests

A flow-field engendered by two crossing shock waves (with slopes $\alpha = \pm 22.23°$) in supersonic uniform flow ($M = 4$) was calculated in the numerical tests. The steady inflow parameters are posed at lateral boundaries. These conditions are valid for a domain where shocks have not reached the boundaries yet. The test problem was chosen due to the availability of an analytical solution.

A first-order finite-difference scheme (donor cells (Roache 1976, Alekseev and Navon 2005a)) was used. The technique proposed here is not limited by the first-order schemes (chosen here for convenience sake) since the adjoint weighted error estimation technique is also well tested for second order schemes (Venditti and Darmofal 2002, Alekseev and Navon 2006).

Figure 1 illustrates the isolines of the density in the flow-field and Figure 2 illustrates the isolines of the adjoint density (the concentration of isolines corresponds to a reference point).

The time evolution starts from a spatially uniform initial guess. During the time relaxation (after every 50 steps), the adjoint problem was solved and the value of the convergence error (41) was estimated. Figure 3 presents the history of the density error (deviation from the analytic value) at the reference point past crossing shocks as a function of time. The difference between the calculated and analytic solution, the convergence error estimated via adjoint parameters and the commonly used convergence indicator $\max_{i,j}|\rho_{ij}^{n+1} - \rho_{ij}^{n}|$ (multiplied by a coefficient 200 for the sake of visibility) are compared in Figure 3.

One can see that the commonly used convergence indicator $\max_{i,j}|\rho_{ij}^{n+1} - \rho_{ij}^{n}|$ provides a qualitatively correct pattern of time evolution but it does not involve any quantitative information for the distance from the exact solution. The adjoint estimation of the convergence error at the initial stage of relaxation deviates significantly from the exact value due to the nonlinearity of the problem.

Figure 4 provides the final part of the temporal evolution. Curve 1 presents the deviation of the numerical solution from the analytic one (shifted by the value of the error caused by the spatial approximation). Curve 2 presents the error estimation using the iteration residual and adjoint parameters. The error of the spatial approximation at the final stage of iteration is marked as line 3. This error was estimated using an adjoint weighted

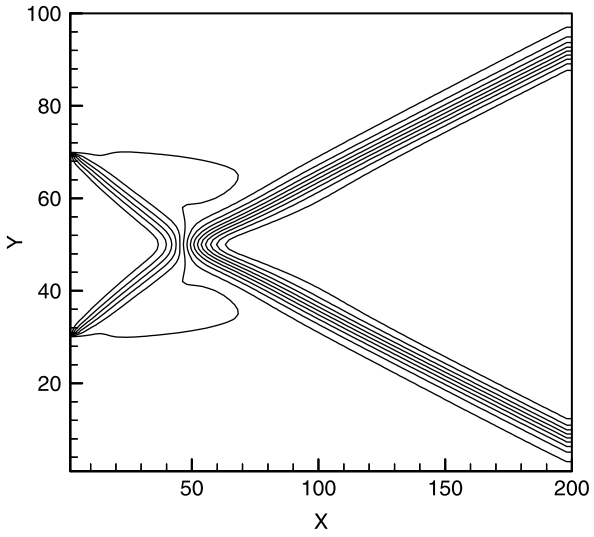

Figure 1. Isolines of the density.

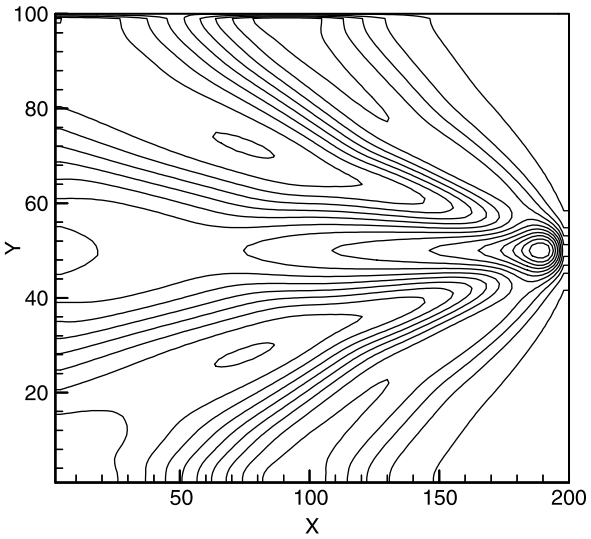

Figure 2. Isolines of the adjoint density.

residual according to Giles and Pierce (1997), Giles and Pierce (1999), Oden and Vemaganti (2000), Giles and Suli (2002), Oden and Prudhomme (2002), Giles *et al.* (2004) and Pierce and Giles (2004). Using the solution of the above adjoint problem the variation of the cost functional was expressed as a function of the truncation error in the following form:

$$\delta\varepsilon = \int_\Omega \int (\delta\rho\Psi_\rho + \delta U\Psi_U + \delta V\Psi_V + \delta e\Psi_e)\mathrm{d}X\,\mathrm{d}Y. \quad (42)$$

Here, $\delta\rho$, etc. are the truncation errors obtained by the Taylor series expansion of finite differences according to Alekseev and Navon (2005a, 2005b). The coefficients in the expansion terms contain derivatives that are calculated from the numerically computed flow-field at the final stage t_f.

For the final stage of the time relaxation, the adjoint error estimation (41) is close to the true convergence error. It should be noted that by increasing the number of iterations the convergence error may be set to be negligibly small compared with the approximation error.

The results presented in Figures 1–4 were calculated on a grid of 100 nodes in the y-direction. Similar results are obtained for grid densities varying between 50 and 200 nodes, i.e. for finer mesh resolution.

5. Discussion

The commonly used convergence indicator $\max_{i,j}|\rho_{ij}^{n+1} - \rho_{ij}^{n}|$ (Roache 1976) qualitatively correctly reflects the convergence but does not provide a quantitative estimation of the deviation from the steady state. On the contrary, the adjoint based convergence indicator provides a quantitative estimate of the iteration error. At the starting stage of the time relaxation, the adjoint error estimation deviates significantly from an exact error due to nonlinearity (this effect is considered also in Alekseev (2006)). At the final stage, the adjoint error estimation is quite close to the exact error (if the spatial approximation error is taken into account). Unfortunately, if the exact solution is unknown, it is difficult to determine when the adjoint estimate approaches the true error. A value of the spatial approximation error (Giles and Pierce 1997, Giles and Pierce 1999, Oden and Vemaganti 2000, Giles and Suli 2002, Oden and Prudhomme 2002, Venditti and Darmofal 2002, Braack and Ern 2003, Cnossen *et al.*

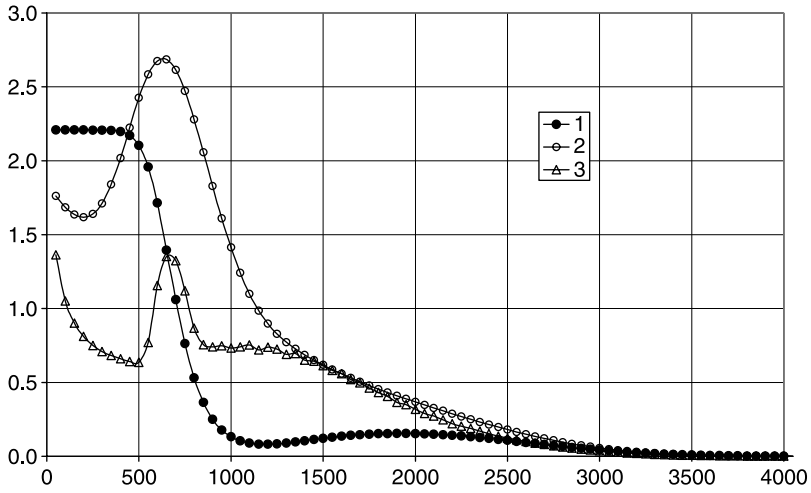

Figure 3. The history of density relaxation as a function of the number of time steps. 1: deviation of the numerical solution from the analytic one; 2: adjoint estimation of the convergence error; 3: convergence indicator $\max_{i,j}|\rho_{ij}^{n+1} - \rho_{ij}^{n}|^{*}200$.

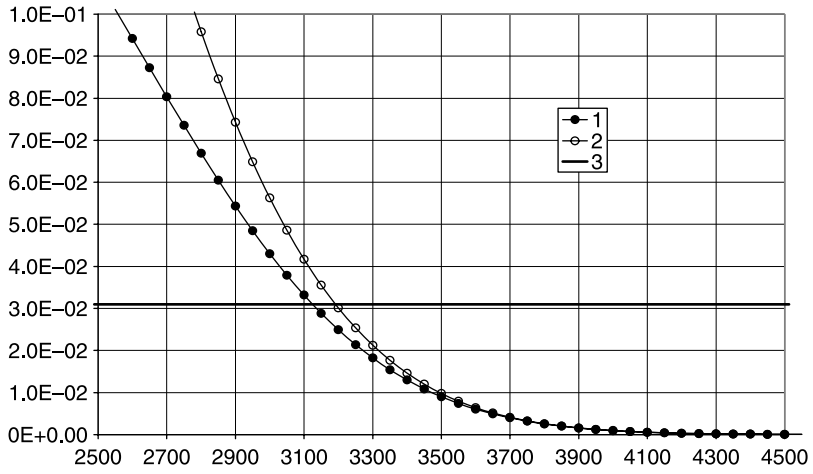

Figure 4. The history of density relaxation (continuation of Figure 3). 1: deviation of the numerical solution from the analytic one shifted by the approximation error; 2: adjoint estimation of the convergence error; and 3: the error of spatial approximation.

2003, Giles *et al.* 2004, Pierce and Giles 2004, Alekseev and Navon 2005a, 2005b, Alekseev and Navon 2006) may serve as an auxiliary criterion for estimating the acceptability of a convergence error. The approximation error may be calculated using already known adjoint parameters and some estimate of the truncation error provided either by the Taylor series or by the action of some postprocessor (Giles and Suli 2002, Alekseev and Navon 2006) on the flow-field. This calculation is based on already prepared fields of direct and adjoint parameters; hence it is not computationally expensive.

In general, the considered estimate may serve for checking the convergence-stopping criterion, if one considers a necessary tolerance of the cost functional for termination of the iterations. However, if we need to track the iterations by estimating the convergence quality according to (41), we should solve the adjoint problem at every checkpoint which implies a large computational burden. Thus, the number of time points, where the estimation is performed, should be limited.

6. Conclusions

The error in the cost functional caused by truncating iterations may be calculated using adjoint variables and the value of the iteration residual.

Numerical tests demonstrate this approach to be able to provide correctly the error of the cost functional (density at a checkpoint) caused by truncating the time iterations for 2D Euler equations.

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