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An estimation of the sensitivity of numerical error norm using adjoint model

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SUMMARY

A posteriori estimation of the numerical error sensitivity to the local truncation error is addressed using adjoint model endowed with the information on the error field. The numerical error is estimated from the solution of the linear tangent model (LTM) or from a Richardson extrapolation. The local truncation error used in the LTM is obtained by the action of a high-order finite-difference stencil on the field computed by the main (low-order accuracy) algorithm. Copyright © 2009 John Wiley & Sons, Ltd.

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KEY WORDS: adjoint problem; *a posteriori* error estimation; numerical error norm; sensitivity

1. INTRODUCTION

At present, there is a body of publications addressing the *a posteriori* estimation of the error of certain important functionals using adjoint equations (see for instance, [1–4]). The adjoint approach provides a fast method for calculation of both the variation of the functional and its sensitivity to errors of different origin including the truncation error. However, the error of practically important functional (pointwise parameters at important locations, integral values, such as drag, lift, average temperature, etc.) provides only a part of information regarding the error of total solution. The global information regarding solution quality is presented by norms of the solution error (perturbation). From the perspective of estimation of the total solution quality, the methods for the estimation of error norms and their sensitivities using adjoint equations are of a significant interest. However, the adjoint equations are only seldom used for the estimation of the error norm and the authors are aware of only several works addressing this topic [5–7] and limited to some particular cases for elliptic equations.

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In the present work we consider the sensitivity of norms of the solution perturbation to a source-like disturbing term. The emphasis is placed on the sensitivity of norms of the numerical error to a local truncation error in application to a finite-difference solution. A special adjoint model endowed by information on the numerical error field is considered. The numerical error is calculated using linear tangent model (LTM) (loaded by the truncation error, i.e. the truncation error is a source in LTM) or by Richardson extrapolation and provides the information necessary for solving this adjoint problem. The estimation of truncation error is obtained using the action of a high-order accuracy finite-difference stencil on a previously computed field. The resulting adjoint state provides the global sensitivity of the numerical error norm to the local truncation error, which constitutes the main feature of the present approach that distinguishes it from other contributions [5–7] related to this topic. This technique is applicable to parabolic and elliptic problems of rather general form restricted only by the requirement for existence of a continuous Gateaux differential.

The present approach to the generation of an adjoint model has certain features in common with the search for the initial conditions providing maximal growth of a perturbation norm as described in References [8–10]. In these works the maximally growing perturbations of the initial state are considered as the dominant singular vectors of the Fisher matrix composed of solutions of the LTM and adjoint problem. This implies solving adjoint problem loaded by the perturbations (i.e. solutions of LTM$-\Delta T$ are a source term in adjoint) taken at the final time. The technique used by Farrell [8, 9] and Buizza *et al.* [10] has essential similarities with the present paper approach. However, it is not adopted for the present paper purposes such as the search of sensitivity or the minimization of error norm.

2. ALGORITHM OUTLINE

Let us briefly consider a formal scheme of the adjoint-based *a posteriori* estimation of the error norm sensitivity. Let the problem of interest (forward one) be governed by the equations

$$\begin{aligned} N(\tilde{T}) &= 0 \quad \text{in } Q \subset R^n \\ B\tilde{T} &= e \quad \text{on } \partial Q \end{aligned} \tag{1}$$

Here $\tilde{T} \in H^k(Q)$ is an exact solution of (1), B is a linear differential operator. Herein we imply $k=1,2$, which corresponds to most CFD problems and $n \geqslant 2$. $H^k(Q)$, $L_2(Q)$, $L_1(Q)$ are standard spaces of functions.

Let N be a nonlinear differential operator ($H^k(Q) \to L_2(Q)$), assumed to possess the following continuous Gateaux differential:

$$N_T(\tilde{T})\Delta T = \lim_{s\to 0} \frac{N(\tilde{T}+s\Delta T) - N(\tilde{T})}{s} \tag{2}$$

Consider now the forward problem perturbed by some source term $s\delta T$ and some boundary term $s\Delta e$.

$$\begin{aligned} N(\tilde{T}+s\Delta T)+s\delta T &= 0 \quad \text{in } Q \subset R^n \\ B(\tilde{T}+s\Delta T) &= e+s\Delta e \quad \text{on } \partial Q \end{aligned} \tag{3}$$

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The Gateaux differential of the forward problem assumes the form

$$N_T(\tilde{T})\Delta T + \delta T = 0, \quad Q \subset R^n$$
$$B\Delta T = \Delta e \quad \text{on } \partial Q \tag{4}$$

It represents the LTM (formally coinciding with the equations for perturbations, but avoiding assumptions on the smallness of ΔT).

We will use the bilinear identity [11, 12]

$$(N_T \Delta T, \Psi)_{L_2(Q)} = (N_T^* \Psi, \Delta T)_{L_2(Q)} + (\Gamma^* \Psi, \Delta T)_{L_2(\partial Q)}$$

where N_T^* is an adjoint operator obtained via integration by parts, Γ^* are the corresponding boundary terms and Ψ is an adjoint variable.

We are interested in the computation of a sensitivity of the error norm

$$\varepsilon = (\|\Delta T\|_{L_2(Q)})^2 = (\Delta T, \Delta T)_{L_2(Q)} \tag{5}$$

to perturbations δT.

Let us introduce the adjoint variables $\varphi \in H^1(Q)$ and formulate the Lagrangian, which is based on LTM (4)

$$L = (\Delta T, \Delta T)_{L_2(Q)} + (N_T(\tilde{T})\Delta T + \delta T, \varphi)_{L_2(Q)} + (B\Delta T - \Delta e, \varphi)_{L_2(\partial Q)} \tag{6}$$

On solving the LTM, the norm of the error is equal to above Lagrangian. It may be rewritten using the bilinear identity as

$$\begin{aligned} L &= (\delta T, \varphi)_{L_2(Q)} + (\Delta T, \Delta T)_{L_2(Q)} + (N_T \Delta T, \varphi)_{L_2(Q)} + (B\Delta T - \Delta e, \varphi)_{L_2(Q)} \\ &= (\delta T, \varphi)_{L_2(Q)} + (N_T^* \varphi, \Delta T)_{L_2(Q)} + (\Gamma^* \varphi, \Delta T)_{L_2(\partial Q)} + (\Delta T, \Delta T)_{L_2(Q)} \\ &\quad + (B^* \varphi, \Delta T)_{L_2(\partial Q)} - (\Delta e, \varphi)_{L_2(\partial Q)} \\ &= (\delta T, \varphi)_{L_2(Q)} + (N_T^* \varphi + \Delta T, \Delta T)_{L_2(Q)} + ((B^* + \Gamma^*)\varphi, \Delta T)_{L_2(\partial Q)} - (\Delta e, \varphi)_{L_2(\partial Q)} \end{aligned} \tag{7}$$

Assuming φ to be a solution of the following adjoint problem:

$$N_T^* \varphi + \Delta T = 0 \quad \text{in } Q$$
$$(B^* + \Gamma^*)\varphi = 0 \quad \text{on } \partial Q \tag{8}$$

we may obtain

$$L = (\delta T, \varphi)_{L_2(Q)} - (\Delta e, \varphi)_{L_2(\partial Q)} \tag{9}$$

If we take $\Delta e = 0$ then

$$\varepsilon = L = (\|\Delta T\|_{L_2(Q)})^2 = \int_\Omega \delta T \varphi \, \mathrm{d}Q \tag{10}$$

Here, the adjoint variable φ has the meaning of sensitivity (weight function) of this norm to a perturbation δT. This sensitivity is the nonlinear ($\varphi = \varphi(\delta T)$) analogue of the Green function.

If we assume $\delta T = 0$ and $\Delta e \neq 0$, we may obtain another form of (9) ($\varepsilon = -(\Delta e, \varphi)_{L_2(\partial Q)}$) and the corresponding adjoint problem. This statement is similar to the one used in References [8–10]

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for the search of maximally growing initial perturbations and is not considered in the present paper.

Herein we consider a special case when δT means the truncation error. Let the numerical solution be provided by the following finite-difference equations:

$$\begin{aligned} N_h T_h &= 0 \quad \text{in } Q \subset R^n \\ B T_h &= e \quad \text{on } \partial Q \end{aligned} \tag{11}$$

As the result of its solution, we obtain a grid function T_h. We assume the existence of a function $T \in C^{\infty}(Q)$ that coincides with the grid function at the nodes (only regular grids are considered). The finite differences in $N_h T$ may be expanded using Taylor series (details of which are provided in the following section). Since the grid is arbitrary, we assume that Equation (11) may be replaced by

$$NT + \delta T_h = 0 \quad \text{in } Q \subset R^n \tag{12}$$

Here δT_h is a formal truncation error containing an infinite number of terms of Taylor expansion. Equation (12) is the differential approximation of a finite-difference scheme [13, 14]. It is an infinite dimensional analogue of the finite-difference scheme and it may be treated as the disturbed initial equation. The corresponding perturbations (numerical error) $\Delta T = T - \tilde{T}$ are governed by the following equations (LTM):

$$\begin{aligned} N_T \Delta T + \delta T_h &= 0 \quad \text{in } Q \subset R^n \\ B \Delta T &= 0 \quad \text{on } \partial Q \end{aligned} \tag{13}$$

Herein, we use the notion of 'differential approximation' of a finite-difference scheme to mean some projection of a discrete (vector) solution back to the infinite-dimensional space. This method consists in some inverse transformation when compared with the standard discretization based on finite differences. Details may be found in [13, 14]. In the following analysis we will consider a finite form of δT_h using the Lagrange remainder, while in numerical tests we obtain δT_h by post processing in accordance with [15, 16].

As a result, Equation (10) assumes the form $(\|\Delta T\|_{L_2(Q)})^2 = \int_Q \delta T_h \varphi \, \mathrm{d}Q$ that provides the nonlinear sensitivity of the numerical error to local truncation error.

3. ESTIMATION OF ERROR NORMS FOR FINITE-DIFFERENCE APPROXIMATION OF HEAT CONDUCTION

3.1. Estimation of L_2 norm of error

Let us consider *a posteriori* estimation of the norm of temperature error $\|\Delta T\|_{L_2}$ for the finite-difference solution of the unsteady one-dimensional heat conduction equation

$$\frac{\partial \tilde{T}}{\partial t} - \lambda \frac{\partial^2 \tilde{T}}{\partial x^2} = 0 \quad \text{in } Q = \Omega \times (0, t_f), \quad \Omega \in R^1 \tag{14}$$

with initial conditions

$$\tilde{T}(0, x) = T_0(x), \qquad T_0(x) \in L_2(\Omega) \tag{15}$$

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and boundary conditions

$$\left.\frac{\partial \tilde{T}}{\partial x}\right|_{x=0}=0, \quad \left.\frac{\partial \tilde{T}}{\partial x}\right|_{x=X}=0 \tag{16}$$

Here λ is thermal diffusion coefficient, $\lambda=$Const, $\tilde{T}$ the temperature (considered here as exact, nonperturbed), x the coordinate, X the thickness, t the time, t_f the duration of the process, Ω the spatial domain of calculation, $\tilde{T}(t,x)\in C^{\infty}(Q)$. In this space the problem is well-posed [17].

We illustrate the above-considered technique for a linear problem due to the availability of a well-known analytical solution used in numerical tests. However, this technique is not restricted to linear problems and may be applied in vicinity of any solution of a nonlinear forward problem, provided that the corresponding Gateaux differential exists.

Consider a finite-difference approximation of Equation (14) of first-order accuracy in time and second-order accuracy in space:

$$\frac{T_k^{n+1}-T_k^n}{\tau}-\lambda\frac{T_{k+1}^{n+1}-2T_k^{n+1}+T_{k-1}^{n+1}}{h^2}=0 \tag{17}$$

Here T_k^n is the solution of the finite-difference equation, τ is the time step and h is the spatial step size. Herein we assume that there exists a smooth function $T(t,x)$ that coincides with T_k^n at all grid points [13]. Let us expand the function at nodes $(T(t_n+\tau,x_k),T(t_n,x_k\pm h))$ in an infinite Taylor series over τ and h and substitute it into (17). Then the following equation may be stated with δT_h denoting the Taylor series residual as

$$\frac{\partial T}{\partial t}-\lambda\frac{\partial^2 T}{\partial x^2}+\delta T_h=0 \tag{18}$$

Since the grid is not specified, this form is considered to be general. So, according to [13, 14], the numerical solution of Equation (17) is considered to be equivalent to solving the perturbed equation (18). On the specified grid the source term δT_h means the local truncation error that can be calculated using a differential approximation of the finite-difference scheme [13, 14] or by the special postprocessor [2, 15, 16]. For the considered finite-difference scheme, the corresponding terms may be represented as the remainder in the Lagrange form:

$$\begin{aligned}
\delta T_h &= \delta T_t+\delta T_x\\
\delta T_t &= \frac{1}{2}\tau\frac{\partial^2 T(t_n+\alpha_k^n\tau,x_k)}{\partial t^2}, \quad t\in(t_n,t_{n+1}), \quad \alpha_k^h\in(0,1)\\
\delta T_x &= -\frac{\lambda}{24}h_k^2\left(\frac{\partial^4 T(t_{n+1},x_k+\beta_k^n h)}{\partial x^4}+\frac{\partial^4 T(t_{n+1},x_k-\gamma_k^n h)}{\partial x^4}\right), \quad x\in(x_{k-1},x_{k+1})\\
&\quad \beta_k^h\in(0,1), \quad \gamma_k^h\in(0,1)
\end{aligned} \tag{19}$$

The transfer of error $\Delta T\,(T=\tilde{T}+\Delta T)$ is determined by LTM

$$\frac{\partial \Delta T}{\partial t}-\lambda\frac{\partial^2 \Delta T}{\partial x^2}+\delta T_h=0 \quad \text{in } Q=\Omega\times(0,t_f), \quad \Omega\in R^1 \tag{20}$$

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with initial conditions

$$\Delta T(0,x)=0 \tag{21}$$

and boundary conditions

$$\left.\frac{\partial \Delta T}{\partial x}\right|_{x=0}=0, \quad \left.\frac{\partial \Delta T}{\partial x}\right|_{x=X}=0 \tag{22}$$

The norm of error $\|\Delta T\|_{L_2(Q)}$ assumes the form:

$$(\|\Delta T\|_{L_2(Q)})^2=\int (\Delta T)^2\,\mathrm{d}x\,\mathrm{d}t \tag{23}$$

Let us now introduce a Lagrangian comprised of the estimated value (23) and the weak statement of LTM (20), which is equal to the norm of error on the solution of the LTM

$$L=\int \Delta T\cdot\Delta T\,\mathrm{d}x\,\mathrm{d}t+\int\left(\frac{\partial \Delta T}{\partial t}-\lambda\frac{\partial^2 \Delta T}{\partial x^2}+\delta T_h\right)\varphi\,\mathrm{d}x\,\mathrm{d}t \tag{24}$$

Using integration by parts

$$L=\int \Delta T\,\Delta T\,\mathrm{d}x\,\mathrm{d}t+\int \delta T_h\varphi\,\mathrm{d}x\,\mathrm{d}t-\int\left(\frac{\partial\varphi}{\partial t}+\lambda\frac{\partial^2\varphi}{\partial x^2}\right)\Delta T\,\mathrm{d}x\,\mathrm{d}t$$

$$+\int \varphi\Delta T\Big|_0^{t_f}\mathrm{d}x+\lambda\int\frac{\partial\varphi}{\partial x}\Delta T\Big|_0^X\mathrm{d}t-\lambda\int\varphi\frac{\partial\Delta T}{\partial x}\Big|_0^X\mathrm{d}t$$

we obtain the following adjoint problem (in deviation from the standard techniques [12], we do not differentiate the goal functional that is essential for derivations of the following subsection):

$$\frac{\partial\varphi}{\partial t}+\lambda\frac{\partial^2\varphi}{\partial x^2}+\Delta T=0 \tag{25}$$

Initial condition:

$$\varphi(t_f,x)=0 \tag{26}$$

and boundary conditions:

$$\left.\frac{\partial\varphi}{\partial x}\right|_{x=0}=0, \quad \left.\frac{\partial\varphi}{\partial x}\right|_{x=X}=0 \tag{27}$$

This problem is solved in the reverse time direction starting from t_f. By solving this problem along with Equations (20)–(22), the norm of error may be expressed as:

$$(\|\Delta T\|_{L_2(Q)})^2=\int \delta T_h\varphi\,\mathrm{d}x\,\mathrm{d}t \tag{28}$$

Certainly, if the truncation error (residual) δT_h is known, one may compute directly the norm of the error by solving Equations (20)–(22). However, the adjoint-based estimation enables the

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determination of sensitivity of this norm to a local truncation error that may be useful for adaptive mesh refinement in problems similar to that considered in [4]. Thus, for sensitivity estimation we need to carry out three calculations: forward model (14), LTM (20), and adjoint model (25). This configuration is more unfavorable when compared with Inverse Problems that usually require only calculation of the forward and adjoint models.

3.2. *Estimation of L_1 norm of error*

For the estimation of error in L_1 norm, the corresponding expressions assume the form:

Numerical error norm

$$\|\Delta T\|_{L_1(Q)} = \int |\Delta T|\, \mathrm{d}x\, \mathrm{d}t = \int \mathrm{sign}(\Delta T)\Delta T\, \mathrm{d}x\, \mathrm{d}t \tag{29}$$

(note that we do not differentiate this functional).

Lagrangian

$$L = \int \mathrm{sign}(\Delta T)\cdot\Delta T\, \mathrm{d}x\, \mathrm{d}t + \int \left(\frac{\partial \Delta T}{\partial t} - \lambda \frac{\partial^2 \Delta T}{\partial x^2} + \delta T_h\right)\varphi_1\, \mathrm{d}x\, \mathrm{d}t \tag{30}$$

Adjoint equation

$$\frac{\partial \varphi_1}{\partial t} + \lambda \frac{\partial^2 \varphi_1}{\partial x^2} + \mathrm{sign}(\Delta T) = 0 \tag{31}$$

with initial condition

$$\varphi_1(t_f, x) = 0 \tag{32}$$

and boundary conditions

$$\left.\frac{\partial \varphi_1}{\partial x}\right|_{x=0} = 0, \quad \left.\frac{\partial \varphi_1}{\partial x}\right|_{x=X} = 0 \tag{33}$$

The norm of the numerical error is:

$$\|\Delta T\|_{L_1} = \int \delta T_h \varphi_1\, \mathrm{d}x\, \mathrm{d}t \tag{34}$$

This expression is similar to (28) although a different adjoint temperature is assumed. In general, the norm $\|\Delta T\|_{L_2}$ is of greater use; however, $\|\Delta T\|_{L_1}$ has some applications that are considered below.

4. NUMERICAL TESTS

The numerical tests are conducted for the well-known analytical solution that readily yields the numerical error as the difference between numerical and analytical solutions. The tests are

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simulating the temperature field evolution generated by a pointwise heat source (where t_0, ξ are the initial time and the coordinate of the point source, respectively)

$$T_{\mathrm{an}}(t,x)=\frac{q}{2\sqrt{\pi\lambda(t-t_0)}}\exp\left(-\frac{(x-\xi)^2}{4\lambda(t-t_0)}\right) \tag{35}$$

We use the data $f_k=T_0(x_k)$ calculated by (35) as the initial data when solving the finite-difference Equation (17). The length X of the spatial interval is chosen so as to provide a negligible effect of the boundary condition compared with the effect of approximation error. The round-off errors were estimated by comparing the calculations obtained with single and double precisions, and the difference was found to be negligible.

The same implicit finite-difference method (implemented using the Thomas algorithm) of second-order accuracy in space and first-order over time (Equation (17)) was applied to solve the heat transfer equation, the LTM, and the adjoint equation. The spatial grid consisted of 100–1000 nodes, the time integration contained 100–10 000 steps. The results obtained are similar within this range of steps. The illustrations, presented herein, have been carried out with 400 spatial nodes and for 400 time steps. Thermal diffusivity was taken as $\lambda=2\cdot 10^{-7}\,\mathrm{m}^2/\mathrm{s}$.

A fourth-order accurate (for both time and space variables) stencil was used for the estimation of the residual:

$$\eta_k^n=\frac{-T_k^{n+2}+8T_k^{n+1}-8f_k^{n-1}+f_k^{n-2}}{12\tau}-\lambda\frac{T_{k+2}^{n+1}+16T_{k+1}^{n+1}-30T_k^{n+1}+16T_{k-1}^{n+1}-T_{k-2}^{n+1}}{12h^2} \tag{36}$$

Using the same approach as that used for deriving Equation (18) we may obtain

$$\eta_k^n=\frac{\partial T}{\partial t}-\lambda\frac{\partial^2 T}{\partial x^2}+\delta T_{x,4}+\delta T_{t,4} \tag{37}$$

Herein we use the function T from Equation (18). The substitution of (18) into (37) yields

$$\eta_k^n=-\delta T_t-\delta T_x+\delta T_{x,4}+\delta T_{t,4}=-\delta T_t-\delta T_x+O(h^4)+O(\tau^4) \tag{38}$$

This expression demonstrates an intuitively transparent result, namely that the residual, engendered by the action of the high-order finite-difference stencil on the solution obtained by the low-order scheme, contains the sum of both schemes' truncation errors. If higher-order terms are neglected, it may be used for the estimation of the truncation error

$$(\delta T_h)_k^n\approx-\eta_k^n \tag{39}$$

This value is used when Equation (20) is solved to obtain numerical error ΔT, which is, at the next step, used in the adjoint problem (25) for calculating the sensitivity.

Solving LTM (20) implies a computational burden caused by coding and debugging an additional problem. As an alternative, the Richardson extrapolation [18] may be used to calculate ΔT. If we have solutions on three different grids (herein, with twofold differences in spatial and temporal steps) and know the orders of convergence (first order in time and second order in space), then

$$\begin{aligned}T_{n,k}^1&=\tilde{T}_{n,k}+C_{t,n,k}\tau+C_{x,n,k}h^2\\T_{n,k}^2&=\tilde{T}_{n,k}+C_{t,n,k}\tau/2+C_{x,n,k}h^2\\T_{n,k}^3&=\tilde{T}_{n,k}+C_{t,n,k}\tau+C_{x,n,k}h^2/4\end{aligned} \tag{40}$$

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It can be easily obtained that

$$\begin{aligned} C_{x,n,k} &= 4(T^1_{n,k} - T^3_{n,k})/(3h^2) \\ C_{t,n,k} &= 2(T^1_{n,k} - T^2_{n,k})/\tau \end{aligned} \tag{41}$$

and

$$\Delta T_{n,k} = T^1_{n,k} - \tilde{T}_{n,k} = C_{t,n,k}\tau + C_{x,n,k}h^2 \tag{42}$$

So we should solve the forward problem (several times) and the adjoint one loaded with the information on ΔT obtained from the Richardson extrapolation. This approach enables one to avoid the coding and debugging of the LTM at the expense of additional runs of the forward problem.

Figure 1 presents a comparison of the temperature error obtained from the LTM, the error estimated from Richardson extrapolation, and the difference between numerical and analytical solutions (true error) as a function of the grid number. One can see that both solving the LTM (20) with a source term assuming the form (36) and the Richardson extrapolation provide an acceptable approximation of the true numerical error.

Figure 2 displays isolines of the temperature field in (x,t) plane, while Figure 3 provides isolines of the truncation error δT calculated by the action of stencil (36) on the temperature field of Figure 2.

Figure 4 presents the field of the numerical error ΔT calculated by linear tangent problem (20) with the truncation error δT provided in Figure 3. The distribution of this error along the x coordinate at the final time is presented in Figure 1. The corresponding adjoint field is displayed in Figure 5.

The results of $\|\Delta T\|_{L_2}$ calculation using Equation (20) as compared with the result obtained using the adjoint equation (28) are presented in Tables I and II for different time steps with a fixed space step (400 and 100 nodes over space). The adjoint field presented in Figure 5 may

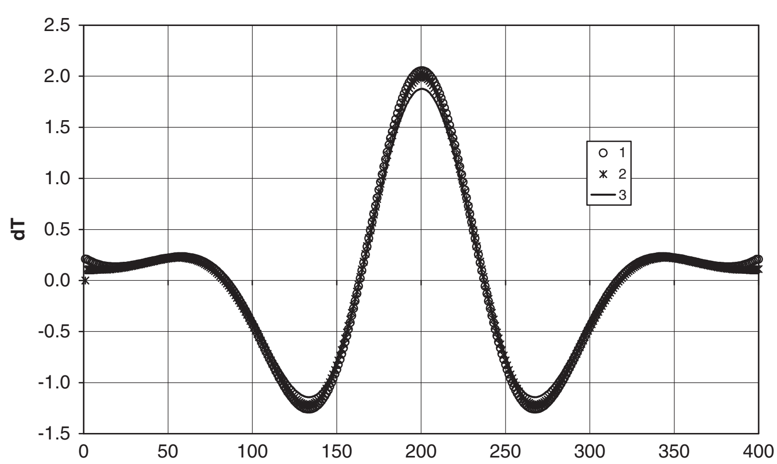

Figure 1. Numerical error ΔT. 1—Linear Tangent Problem; 2—Richardson extrapolation; and 3—difference between numerical and analytical solutions.

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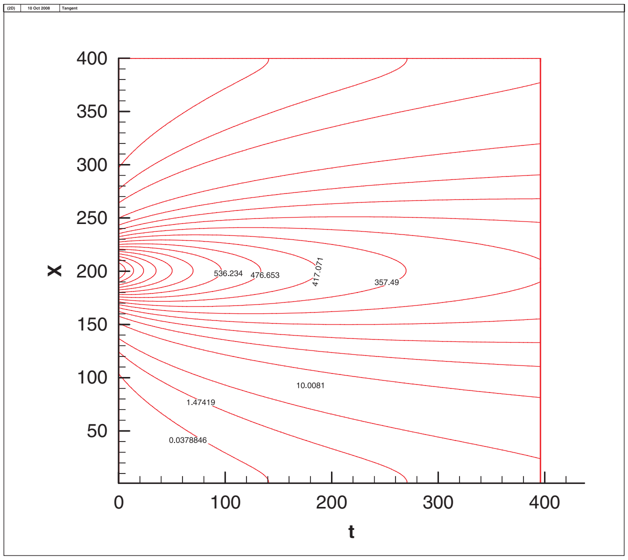

Figure 2. Temperature field.

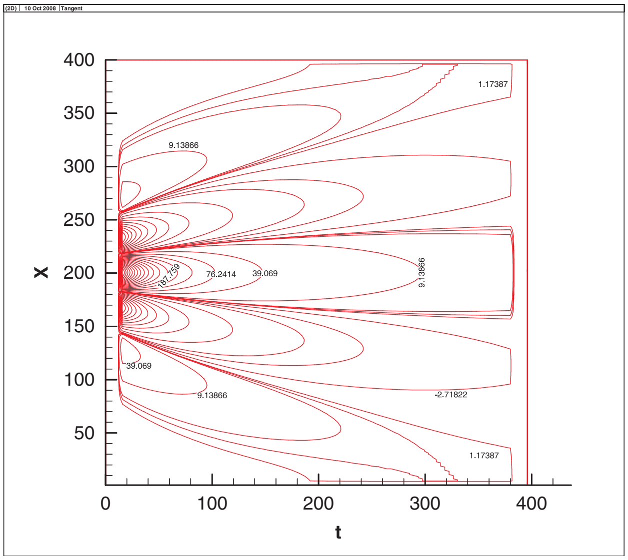

Figure 3. Truncation error δT calculated by action of stencil (36).

be considered as the sensitivity of this norm to the local truncation error, a result that provides significant additional information when compared with the LTM solution, which provides only field of ΔT and $\|\Delta T\|_{L_2}$, while our approach provides the means to diminish ΔT by affecting δT. This provides us with additional options when compared with the pure LTM.

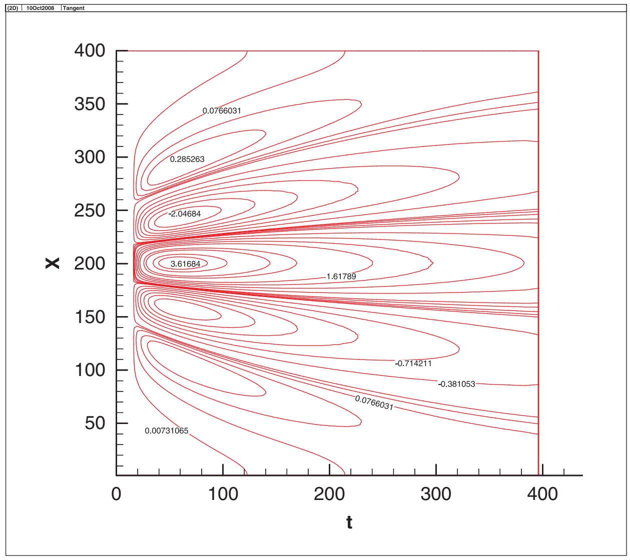

Figure 4. Numerical error ΔT field.

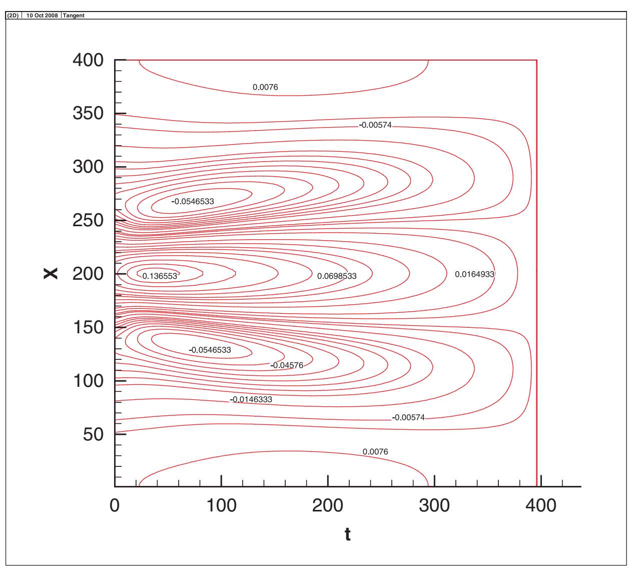

Figure 5. Adjoint temperature φ field from Equation (25) for $\|\Delta T\|_{L_2}$.

Similar calculations are conducted for the adjoint equation (31) aimed at estimating $\|\Delta T\|_{L_1}$. The corresponding field of the adjoint temperature is presented in Figure 6. The results of the $\|\Delta T\|_{L_1}$ calculation using Equation (20) compared with the result obtained using the adjoint equation (31)

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Table I. Comparison of numerical error norm $\|\Delta T\|_{L_2}$ computed using the adjoint equation and LTM as a function of time steps (400 spatial nodes).

τ	1.0	0.5	0.1	0.05	0.01
$\|\Delta T\|_{L_2}$, adjoint, Equation (28)	1.6014	0.93973	0.23099	0.12684	0.040510
$\|\Delta T\|_{L_2}$, LTM, Equation (20)	1.6003	0.93941	0.23097	0.12685	0.040509

Table II. Comparison of numerical error norm $\|\Delta T\|_{L_2}$ computed using the adjoint equation and LTM as a function of time steps (100 spatial nodes).

τ	1.0	0.5	0.1	0.05	0.01
$\|\Delta T\|_{L_2}$, adjoint, Equation (28)	3.26259	1.37880	0.24758	0.15859	0.099797
$\|\Delta T\|_{L_2}$, LTM, Equation (20)	3.25823	1.37788	0.24753	0.15857	0.099795

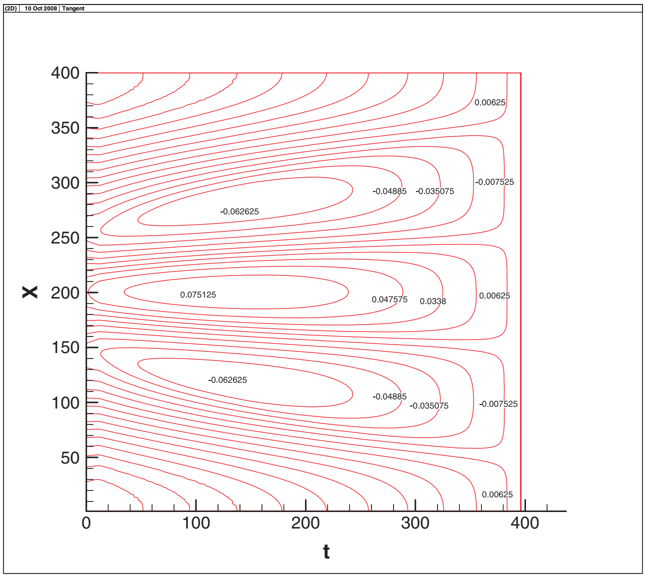

Figure 6. Adjoint temperature φ_1 field from Equation (31) for $\|\Delta T\|_{L_1}$.

are presented in Tables III and IV for different time steps with a fixed space step (400 and 100 nodes over space).

The results provided in Tables I–IV demonstrate a good correlation of solutions, obtained by LTM and adjoint approaches of both statements (25)–(27) and (31)–(33) when the time step τ varies over a range of two orders of magnitude. The presented results verify the solution for both the LTM and the adjoint problems.

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Table III. Comparison of numerical error norm $\|\Delta T\|_{L_1}$ computed using adjoint equation and LTM as a function of time step size (400 spatial nodes).

τ	1.0	0.5	0.1	0.05	0.01
$\|\Delta T\|_{L_1}$, adjoint, Equation (31)	2.0018	1.1470	0.27343	0.14942	0.047526
$\|\Delta T\|_{L_1}$, LTM, Equation (20)	1.9896	1.1437	0.27328	0.14938	0.047523

Table IV. Comparison of numerical error norm $\|\Delta T\|_{L_1}$ computed using adjoint equation and LTM as a function of time step size (100 spatial nodes).

τ	1.0	0.5	0.1	0.05	0.01
$\|\Delta T\|_{L_1}$, adjoint, Equation (31)	2.24599	1.42645	0.58693	0.46768	0.36968
$\|\Delta T\|_{L_1}$, LTM, Equation (20)	2.23255	1.42245	0.58661	0.46756	0.36966

5. DISCUSSION

The sensitivity of the norm of solution perturbation to local errors of different origin exhibits the same form for a significant number of statements. For example, if an uncertainty is connected with the thermal diffusion coefficient, we obtain the form

$$(\|\Delta T\|_{L_2(Q)})^2 = \int_Q \delta\lambda \frac{\partial^2 T}{\partial x^2} \varphi \, \mathrm{d}Q \tag{43}$$

This information may serve to guide experiments aimed at estimation of λ.

For the case of the adaptive mesh refinement we may readily obtain ΔT; nevertheless, the grid cannot just simply be refined for zones of high ΔT^2 since ΔT is a nonlocal value and is engendered by both the local truncation error δT and the transfer of ΔT from other parts of the computational domain. The mesh may however be refined in zones of large $|\delta T \varphi|$, which correctly represents the numerical error transfer. However, due to the nonlinearity of the problem, this mesh adaptation should assume an iterative structure. Additionally, to avoid problems with multidimensional interpolation, we may consider the mesh to be regular at the initial stage and to be regular in a finite number of subdomains after refinement. Adaptive mesh refinement (cubic volumes being divided in smaller cubes) [19] may be considered as an illustration of this approach.

The estimations of norms in L_2 and L_1 are very close in form and are practically identical from a computational cost consideration, although the L_2 norm is commonly used due to natural links with the evaluation of dispersion (for example as in (43)). However, one may see that problem (31) for estimation in L_1 norm uses a lesser amount of information about ΔT (only the sign), when compared with (25). The problem (25) may be recast as $A \cdot \varphi_1 = F$ with a formal solution $\varphi_1 = A^{-1} F$. The norm of the error may be estimated via $\|\Delta T\|_{L_1} = (\varphi_1, \delta T_h)_{L_2} \leqslant \cdot \|\varphi\| \cdot \|\delta T_h\| \leqslant \|A^{-1}\| \cdot \|F\| \cdot \|\delta T_h\|$. For the L_1 norm, an estimate $\|F\|_{L_\infty} \equiv 1$ may be easily obtained that stimulates certain interest in this norm. If we have estimates for $\|A^{-1}\|$ (Reference [5] provides such estimates for some elliptic problems), the upper bound of $\|\Delta T\|_{L_1}$ may be readily obtained from the truncation error without solving an adjoint problem.

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6. CONCLUSION

The sensitivity of the norm of the numerical error to the truncation error may be estimated using a special adjoint model. This model should account for the information concerning numerical error that may be obtained either by solving the tangent linear model or by a Richardson extrapolation. The validity of this approach is confirmed by results of numerical tests for the heat transfer equation solved by a finite-difference discretization.

This sensitivity provides the possibility for carrying out adaptive mesh refinement based on the minimization of the numerical error norm.

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