Lectures - Week 4
Matrix norms, Conditioning, Vector Spaces, Linear Independence, Spanning sets and Basis, Null space and Range of a Matrix

Matrix Norms

Now we turn to associating a number to each matrix. We could choose our norms analogous to the way we did for vector norms; e.g., we could associate the number $\max_{ij} |a_{ij}|$. However, this is actually not very useful because remember our goal is to study linear systems $A\vec{x} = \vec{b}$.

The general definition of a matrix norm is a map from all $m \times n$ matrices to $\mathbf{R}^1$ which satisfies certain properties. However, the most useful matrix norms are those that are generated by a vector norm; again the reason for this is that we want to solve $A\vec{x} = \vec{b}$ so if we take the norm of both sides of the equation it is a vector norm and on the left hand side we have the norm of a matrix times a vector.

We will define an *induced matrix norm* as the largest amount any vector is magnified when multiplied by that matrix, i.e.,

$$\|A\| = \max_{\substack{\vec{x} \in \mathbf{R}^n \\ \vec{x} \neq 0}} \frac{\|A\vec{x}\|}{\|\vec{x}\|}$$

Note that all norms on the right hand side are vector norms. We will denote a vector and matrix norm using the same notation; the difference should be clear from the argument. We say that the vector norm on the right hand side induces the matrix norm on the left. Note that sometimes the definition is written in an equivalent way as

$$\|A\| = \sup_{\substack{\vec{x} \in \mathbf{R}^n \\ \vec{x} \neq 0}} \frac{\|A\vec{x}\|}{\|\vec{x}\|}$$

Example What is $\|I\|$? Clearly it is just one.

The problem with the definition is that it doesn't tell us how to compute a matrix norm for a general matrix A. The following theorem gives us a way to calculate matrix norms induced by the ℓ_∞ and ℓ_1 norms; the matrix norm induced by ℓ_2 norm will be addressed later after we have introduced eigenvalues.

Theorem *Let A be an $m \times n$ matrix. Then*

$$\|A\|_\infty = \max_{1 \leq i \leq m} \Big[\sum_{j=1}^{n} |a_{ij}| \Big] \quad \text{(max absolute row sum)}$$

$$\|A\|_1 = \max_{1 \leq j \leq n} \Big[\sum_{i=1}^{m} |a_{ij}| \Big] \quad \text{(max absolute column sum)}$$

Example Determine $\|A\|_\infty$ and $\|A\|_1$ where

$$A = \begin{pmatrix} 1 & 2 & -4 \\ 3 & 0 & 12 \\ -20 & -1 & 2 \end{pmatrix}$$

We have

$$\|A\|_1 = \max\{(1+3+20), (2+1), (4+12+2)\} = \max\{24, 3, 18\} = 24$$

$$\|A\|_\infty = \max\{(1+2+4), (3+12), (20+1+2)\} = \max\{7, 15, 23\} = 23$$

Proof We will prove that $\|A\|_\infty$ is the maximum row sum (in absolute value). We will do this by proving that

$$\|A\|_\infty \le \max_{1\le i\le m} \Big[\sum_{j=1}^n |a_{ij}| \Big] \quad \text{and then showing} \quad \|A\|_\infty \ge \max_{1\le i\le m} \Big[\sum_{j=1}^n |a_{ij}| \Big]$$

First recall that if $A\vec{x} = \vec{b}$ then

$$b_i = \sum_{j=1}^n a_{ij}x_j \Rightarrow \|\vec{b}\|_\infty = \max_i |b_i| = \max_i |\sum_{j=1}^n a_{ij}x_j|$$

For the first inequality we know that by definition

$$\|A\|_\infty = \max_{\substack{\vec{x}\in \mathrm{R}^n \\ \vec{x}\ne 0}} \frac{\|A\vec{x}\|_\infty}{\|\vec{x}\|_\infty}$$

Now lets simplify the numerator to get

$$\|A\vec{x}\|_\infty = \max_i |\sum_{j=1}^n a_{ij}x_j| \le \max_i \sum_{j=1}^n |a_{ij}|\ |x_j| \le \|\vec{x}\|_\infty \max_i \sum_{j=1}^n |a_{ij}|$$

Thus the ratio reduces to

$$\frac{\|A\vec{x}\|_\infty}{\|\vec{x}\|_\infty} \le \frac{\|\vec{x}\|_\infty \max_i \sum_{j=1}^n |a_{ij}|}{\|\vec{x}\|_\infty} = \max_i \sum_{j=1}^n |a_{ij}|$$

and hence

$$\|A\|_\infty = \max_{\substack{\vec{x}\in \mathrm{R}^n \\ \vec{x}\ne 0}} \frac{\|A\vec{x}\|_\infty}{\|\vec{x}\|_\infty} \le \max_i \sum_{j=1}^n |a_{ij}|\,.$$

Now for the second inequality we know that

$$\|A\|_\infty \ge \frac{\|A\vec{y}\|_\infty}{\|\vec{y}\|_\infty}$$

for any $\vec{y} \in \mathrm{R}^n$ because equality in the definition holds here for the maximum of this ratio. So now we will choose a particular $\vec{y}$ and we will construct it so that it has $\|\vec{y}\|_\infty = 1$. First let p be the row where A has its maximum row sum (or there are two rows, take the first), i.e.,

$$\max_i \sum_{j=1}^n |a_{ij}| = \sum_{j=1}^n |a_{pj}|$$

Now we will take the entries of $\vec{y}$ to be ± 1 so its infinity norm is one. Specifically we choose

$$y_i = \begin{cases} 1 & \text{if } a_{pj} \geq 0 \\ -1 & \text{if } a_{pj} < 0 \end{cases}$$

Defining $\vec{y}$ in this way means that $a_{ij} y_j = |a_{ij}|$. Using this and the fact that $\|\vec{y}\|_\infty = 1$ we have

$$\|A\|_\infty \geq \frac{\|A\vec{y}\|_\infty}{\|\vec{y}\|_\infty} = \max_i |\sum_{j=1}^n a_{ij} y_j| \geq |\sum_{j=1}^n a_{pj} y_j| = |\sum_{j=1}^n |a_{pj}|\,| = \sum_{j=1}^n |a_{pj}|$$

but the last quantity on the right is must the maximum row sum and the proof is complete.

What are the properties that any matrix norm (and thus an induced norm) satisfy? You will recognize most of them as being analogous to the properties of a vector norm.

(i) $\|A\| \geq 0$ and $= 0$ only if $a_{ij} = 0$ for all i, j.
(ii) $\|kA\| = |k|\|A\|$ for scalars k
(iii) $\|AB\| \leq \|A\|\|B\|$
(iv) $\|A + B\| \leq \|A\| + \|B\|$

A very useful inequality is

$$\|A\vec{x}\| \leq \|A\|\|\vec{x}\| \quad \text{for any induced norm}$$

Why is this true?

$$\|A\| = \max_{\substack{\vec{x} \in \mathrm{R}^n \\ \vec{x} \neq 0}} \frac{\|A\vec{x}\|}{\|\vec{x}\|} \geq \frac{\|A\vec{x}\|}{\|\vec{x}\|} \Rightarrow \|A\vec{x}\| \leq \|A\|\|\vec{x}\|$$

The Condition Number of a Matrix

We said one of our goals was to determine if small changes in our data of a linear system produces small changes in the solution. Now lets assume we want to solve $A\vec{x} = \vec{b}$, $\vec{b} \neq \vec{0}$ but instead we solve

$$A\vec{y} = \vec{b} + \vec{\delta b}$$

that is, we have perturbed the right hand side by a small about $\vec{\delta b}$. We assume that A is invertible, i.e., A^{-1} exists. For simplicity, we have not perturbed the coefficient matrix

A. What we want to see is how much $\vec{y}$ differs from $\vec{x}$. Lets write $\vec{y}$ as $\vec{x} + \vec{\delta x}$ and so our change in the solution will be $\vec{\delta x}$. The two systems are

$$A\vec{x} = \vec{b} \qquad A(\vec{x} + \vec{\delta x}) = \vec{b} + \vec{\delta b}$$

What we would like to get is an estimate for the relative change in the solution, i.e.,

$$\frac{\|\vec{\delta x}\|}{\|\vec{x}\|}$$

in terms of the relative change in $\vec{b}$ where $\|\cdot\|$ denotes any induced vector norm. Subtracting these two equations gives

$$A\vec{\delta x} = \vec{\delta b} \quad \text{which implies} \quad \vec{\delta x} = A^{-1}\vec{\delta b}$$

Now we take the (vector) norm of both sides of the equation and then use our favorite inequality above

$$\|\vec{\delta x}\| = \|A^{-1}\vec{\delta b}\| \leq \|A^{-1}\| \, \|\vec{\delta b}\|$$

Remember our goal is to get an estimate for the relative change in the solution so we have a bound for the change. What we need is a bound for the relative change. Because $A\vec{x} = \vec{b}$ we have

$$\|A\vec{x}\| = \|\vec{b}\| \Rightarrow \|\vec{b}\| \leq \|A\|\|\vec{x}\| \Rightarrow \frac{1}{\|\vec{b}\|} \geq \frac{1}{\|A\|\|\vec{x}\|}$$

Now we see that if we divide our previous result for $\|\vec{\delta x}\|$ by $\|A\|\|\vec{x}\| > 0$ we can use this result to introduce $\|\vec{b}\|$ in the denominator. We have

$$\frac{\|\vec{\delta x}\|}{\|A\|\|\vec{x}\|} \leq \frac{\|A^{-1}\|\|\vec{\delta b}\|}{\|A\|\|\vec{x}\|} \leq \frac{\|A^{-1}\|\|\vec{\delta b}\|}{\|\vec{b}\|}$$

Multiplying by $\|A\|$ gives the desired result

$$\frac{\|\vec{\delta x}\|}{\|\vec{x}\|} \leq \|A\|\|A^{-1}\|\frac{\|\vec{\delta b}\|}{\|\vec{b}\|}$$

If the quantity $\|A\|\|A^{-1}\|$ is small, then this means small relative changes in $\vec{b}$ result in small relative changes in the solution but if it is large, we could have a large relative change in the solution.

Definition The *condition number* of a square matrix A is defined as

$$\mathcal{K}(A) \equiv \|A\|\|A^{-1}\|$$

Note that the condition number depends on what norm you are using. We say that a matrix is *well-conditioned* if $\mathcal{K}(A)$ is "small" and *ill-conditioned* otherwise.

Example Find the condition number of the identity matrix using the infinity norm.

Clearly it is one because the inverse of the identity matrix is itself.

Example Find the condition number for each of the following matrices using the infinity norm.. Explain geometrically why you think this is happening.

$$A_1 = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} \qquad A_2 = \begin{pmatrix} 1 & 2 \\ -0.998 & -2 \end{pmatrix}$$

First we need to find the inverse of each matrix and then take the norms. Note the following "trick" for taking the inverse of a 2×2 matrix

$$A_1^{-1} = \frac{-1}{5} \begin{pmatrix} 3 & -2 \\ -4 & 1 \end{pmatrix} \qquad A_2^{-1} = \begin{pmatrix} 500 & 500 \\ -249.5 & -250 \end{pmatrix}$$

Now

$$\mathcal{K}_\infty(A_1) = (7)(1) = 7 \qquad \mathcal{K}_\infty(A_2) = (3)(1000) = 3000$$

Example Calculate the $\mathcal{K}_\infty(A)$ where A is

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and comment on when the condition number will be large.

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

So

$$\mathcal{K}_\infty(A) = \max\{|a| + |b|, |c| + |d|\} \ \max\{\frac{1}{ad - bc}\big(|d| + |b|\big), \frac{1}{ad - bc}\big(|c| + |a|\big)\}$$

Consequently if the determinant $ad - bc \approx 0$ then the condition number can be quite large; i.e., when the matrix is almost not invertible, i.e., almost singular.

A classic example of an ill conditioned matrix is the Hilbert matrix which we have already encountered Here are some of the condition numbers (using the matrix norm induced by the ℓ_2 vector norm).

n	approximate $\mathcal{K}_2(A)$
2	19
3	524
4	15,514
5	476,607
6	$1.495\ 10^7$

Vector (or Linear) Spaces

What we want to do now is take the concept of vectors and the properties of Euclidean space $\mathbf{R}^n$ and generalize them to a collection of objects with two operations defined, addition and scalar multiplication. We want to define these operations so that the usual properties hold, i.e., $x+y=y+x$, $k(x+y)=kx+ky$ for objects other than vectors, such as matrices, continuous functions, etc. We want to be able to add two elements of our set and get another element of the set.

Definition A *vector or linear space* V is a set of objects, which we will call vectors, for which addition and scalar multiplication are defined and satisfy the following properties.

(i) $x+y=y+x$
(ii) $x+(y+z)=(x+y)+z$
(iii) there exists a zero element $0 \in V$ such that $x+0=0+x=x$
(iv) for each $x \in V$ there exists $-x \in V$ such that $x+(-x)=0$
(v) $1x=x$
(vi) $(\alpha+\beta)x=\alpha x+\beta x$, for scalars α, β
(vii) $\alpha(x+y)=\alpha x+\alpha y$
(viii) $(\alpha\beta)x=\alpha(\beta x)$

Example $\mathbf{R}^n$ is a vector space with the usual operation of addition and scalar multiplication.

Example The set of all $m \times n$ matrices with the usual definition of addition and scalar multiplication forms a vector space.

Example All polynomials of degree less than or equal to two on $[-1,1]$ form a vector space with usual definition of addition and scalar multiplication. Here $p(x)=a_0+a_1x+a_2x^2$

Example All continuous functions defined on $[0,1]$.

Note that if $v, w \in V$ then $v+w \in V$; we say that V is *closed under addition.* Also if $v \in V$ and k is a scalar, $kv \in V$ and it is *closed under scalar multiplication.*

Definition A *subspace* S of a vector space V is a nonempty subset of V such that if we take any linear combination of vectors in S it is also in S.

Example A line is a subspace of $\mathbf{R}^2$ because if we take add two points on the line, the result is still a point on the line and if we multiply any point on the line by a constant then it is still on the line.

Example All lower triangular matrices form a subspace of the vector space of all square matrices.

Example Is the set of all points on the given lines a subspace of $\mathbf{R}^2$?

$$y = x \qquad y = 2x + 1$$

The first forms a subspace but the second does not. It is not closed under addition or scalar multiplication, e.g.,

$$2(x, 2x + 1) = (2x, 4x + 2) \quad \text{which is not on the line } y = 2x + 1$$

Linear independence, spanning sets and basis

We now want to investigate how to characterize a vector space, if possible. To do this, we need the concepts of

(i) linear independence/dependence of vectors
(ii) a spanning set
(iii) the dimension of a vector space
(iv) the basis of a finite dimensional vector space

In $\mathbf{R}^2$ the vectors $\vec{v}_1 = (1, 0)^T$ and $\vec{v}_2 = (0, 1)^T$ are special because

(i) every other vector in $\mathbf{R}^2$ can be written as a linear combination of these two vectors, e.g., if $\vec{x} = (x_1, x_2)^T$ then $\vec{x} = x_1\vec{v}_1 + x_2\vec{v}_2$.
(ii) the vectors $\vec{v}_1, \vec{v}_2$ are different in the sense that the only way we can combine them and get the zero vector is with zero coefficients, i.e.,

$$C_1\vec{v}_1 + C_2\vec{v}_2 = \vec{0} \Rightarrow C_1 = C_2 = 0$$

Of course there are many other sets of two vectors in $\mathbf{R}^2$ that have the same properties.

Definition Let V be a vector space. Then the set of vectors $\{\vec{v}_i\}$, $i = 1, \ldots, n$

(i) *span* V if any $\vec{w} \in V$ can be written as a linear combination of the $\vec{v}_i$, i.e.,

$$\vec{w} = \sum_{j=1}^{n} C_j\vec{v}_j$$

(ii) are *linearly independent* if

$$\sum_{j=1}^{n} C_j\vec{v}_j = \vec{0} \quad \Rightarrow \quad C_i = 0, \forall i$$

otherwise they are *linearly dependent.* Note that this says the only way we can combine the vectors and get the zero vector is if all coefficients are zero.

(iii) form a *basis* for V if they are linearly independent and span V.

Example Let V be the vector space of all polynomials of degree less than or equal to two on $[0, 1]$. Then the set of vectors (i.e., polynomials) in V $\{1, x, x^2, 2 + 6x\}$ span V but they are NOT linearly independent because

$$2(1) + 6(x) + 0(x^2) - (2 + 6x) = 0$$

and thus they do not form a basis for V. The set of vectors $\{1, x\}$ are linearly independent but they do not span V because, e.g., $x^2 \in V$ and it can't be written as a linear combination of $\{1, x\}$.

Definition The number of elements in the basis for V is called the *dimension* of V. It is the smallest number of vectors needed to completely characterize V, i.e., that span V.

Example Find a basis for (i) the vector space of all 2×2 matrices and (ii) the vector space of all 2×2 symmetric matrices. Give the dimension.

A basis for all 2×2 matrices consists of the four matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

and so its dimension is 4. A basis for all 2×2 symmetric matrices consists of the three matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and so its dimension is 2.

Four Important Spaces

1. The **column space** (or equivalently the **range**) of A, where A is $m \times n$ matrix is all linear combinations of the columns of A. We denote this by $\mathcal{R}(A)$.

By definition it is a subspace of $\mathbf{R}^m$.

Example Consider the overdetermined system

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

We know that this is solvable if there exists x, y such that

$$x \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix} + y \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

which says that $\vec{b}$ has to be in the column space (or range) of A.

This says that an equivalent statement to $A\vec{x} = \vec{b}$ being solvable is that $\vec{b}$ is in the range or column space of A.

2. The **null space** of A, denoted $\mathcal{N}(A)$, where A is $m \times n$ matrix is the set of all vectors $\vec{z} \in \mathrm{R}^n$ such that $A\vec{z} = \vec{0}$.

Example Find the null space of each matrix

$$A_1 = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \quad A_2 = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \quad A_3 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

For A_1 we have that $\mathcal{N}(A_1) = \vec{0}$ because the matrix is invertible. To see this we could take the determinant or perform GE and get the result.

For A_2 we see that $\mathcal{N}(A_2)$ is $\alpha(-2x_2, x_2)^T$, i.e., the all points on the line through the origin $y = -.5x$. To see this consider GE for the system

$$\begin{pmatrix} 1 & 2 & \\ 2 & 4 & \end{pmatrix} \to \begin{pmatrix} 1 & 2 & \\ 0 & 0 & \end{pmatrix} \Rightarrow 0 \cdot x_2 = 0, \quad x_1 + 2x_2 = 0$$

This says that x_2 is arbitrary and $x_1 = -2x_2$.

For A_3 we see that $\mathcal{N}(A_2)$ is all of R^2.

Example What are the possible null spaces of a 3×3 matrix?

For an invertible matrix it is the (i) zero vector in $\mathbf{R}^3$, we could have (ii) a line through the origin, (iii) a plane through the origin or all of (iv) $\mathbf{R}^3$.