ISC 5935 - Computational Tools for Finite Elements
Homework #1
Assigned 3 September 2014

1. Consider the two-point boundary value problem (BVP):

$$\begin{aligned} -u'' + u =& x, 0 < x < 1, \\ u'(0) =& 2, \\ u(1) =& 0. \end{aligned}$$

- Write down a weak formulation for the BVP; at this point, you do not have to be specific about the underlying spaces.
- Show that any solution of the classical problem is also a solution of the weak formulation of the problem.
- Show that, if a solution of the weak formulation of the problem is sufficiently smooth, it is also a solution of the classical problem.

2. In the boundary value problem given in problem 1, which boundary condition is essential, and which is natural. Why?

3. Consider the piecewise linear function $\psi(x)$ given by:

$$\psi(x) = \begin{cases} 8x & 0.00 \leq x \leq 0.25 \\ -2x + 2.5 & 0.25 \leq x \leq 0.50 \\ -4x + 3.5 & 0.50 \leq x \leq 0.75 \\ 6x - 4 & 0.75 \leq x \leq 1.00 \end{cases}$$

Write $\psi(x)$ as a linear combination of the standard "hat" basis functions on the given partition of [0,1].

4. Show that the entries of the stiffness matrix K^h and the right hand side vector $\vec{b}^h$ are given by (1.18) in the text, provided the basis functions defined by (1.10) are used and the midpoint rule is employed to evaluate the integrals.

5. Let V be a Hilbert space with norm $\| \cdot \|$ and let $f, g \in V$. Verify the *parallelogram law*:

$$\|f + g\|^2 + \|f - g\|^2 = 2\|f\|^2 + 2\|g\|^2 .$$

Note that the name comes from the special case in R^2 where we know that the sum of the squares of the sides of a parallelogram is equal to the sum of the squares of the diagonals.

6. We have defined the standard "hat" basis functions for continuous piecewise linear polynomials which have support (i.e., they are nonzero) over two subintervals (except at the boundaries). Explain why you can't have a basis which has support only over one subinterval.

7. Consider the function $u(x) = x^3 \sin \pi x$ on $[0, 1]$. We want to determine the *best approximation* in the L^2-norm, $\tilde{u}(x)$, to $u(x)$ out of the space of continuous piecewise linear functions which are zero at $x = 0$ and $x = 1$.

- Choose a uniform partition of $[0,1]$ with $h = 0.25$. Write a code to determine the best approximation $\tilde{u}$ to $u(x)$ using the standard "hat" basis functions for continuous piecewise linears. For the integration, use a two-point Gauss quadrature rule. Write your code so that you have a separate function or subroutine which evaluates a basis function at any given point.
- Repeat with $h = 0.125$ and $h = .0625$. For each value of h determine the L^2 norm of the error in $u(x)$ and $\tilde{u}(x)$ (i.e., $\left[\int_0^1 (u - \tilde{u})^2 \, dx\right]^{1/2}$); calculate a numerical rate of convergence (i.e., determine k such that the error is $\mathcal{O}(h^k)$ based upon your three calculations. When calculating the error, apply the two-point Gauss quadrature rule over each subinterval to compute the integral over the entire interval.