FEM Problems in Preliminary Exams

1 Spring 2008

Given the function $f(x, y)$, consider the problem:

$$\begin{cases} -\dfrac{\partial^2 u}{\partial x^2} - \dfrac{\partial^2 u}{\partial y^2} = f(x, y), & \text{for } 0 < x, y < 1 \\ u(x, 0) = u(x, 1) = 0, & \text{for } 0 \le x \le 1 \\ u(0, y) = u(1, y) = 0, & \text{for } 0 \le y \le 1 \end{cases} \tag{1}$$

a. Discuss how you would determine an approximate solution of this problem using a piecewise-linear finite element method.

b. Discuss the factors that affect the accuracy of finite element methods for the approximate solution of this problem.

Solution:

a. Define $\Omega = (0, 1) \times (0, 1), \Gamma = \partial\Omega$, seek $u \in H_0^1(\Omega) \equiv \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma\}$, such that

$$\begin{aligned} (f, v) \equiv \int_\Omega f v \mathrm{d}V &= -\int_\Omega \Delta u\, v \mathrm{d}V \xlongequal{\text{Green' formula}} \int_\Omega \nabla u \cdot \nabla v \mathrm{d}V - \int_\Gamma \frac{\partial u}{\partial n} v \mathrm{d}S \\ &= \int_\Omega \nabla u \cdot \nabla v \mathrm{d}V \equiv A(u, v). \quad \forall v \in H_0^1(\Omega) \end{aligned} \tag{2}$$

In order to determine an approximation solution, we need to construct a finite dimensional subspace $S_0^h(\Omega) \subset H_0^1(\Omega)$:

First make a triangulation of Ω, by subdividing Ω into a set $T^h = \{K_1, ..., K_m\}$ of non-overlapping triangles K_i,

$$\Omega = \bigcup_{K \subset T^h} K = K_1 \cup K_2 \cup ... \cup K_m,$$

such that no vertex of one triangle lies on the edge of another triangle.

Define S_0^h as follows:

$$S_0^h(\Omega) = \{v : v \text{ is continuous on } \Omega, v|_K \text{ is linear for } K \in T_h, v = 0 \text{ on } \Gamma\}.$$

Here, $v|_K$ denotes the restriction of v to K. The subspace $S_0^h(\Omega)$ consists of all continuous functions that are linear on each triangle K and vanish on Γ. The corresponding basis functions $\varphi_j \in S_0^h(\Omega),\ \ j = 1, ..., M$ are then defined by: (See Figure 1)

$$\varphi_j(N_i) = \delta_{ij} \equiv \begin{cases} 1, & i = j; \\ 0, & i \neq j. \end{cases}$$

Here $N_i,\ \ i = 1, ..., M$ are inner nodes of domain Ω.

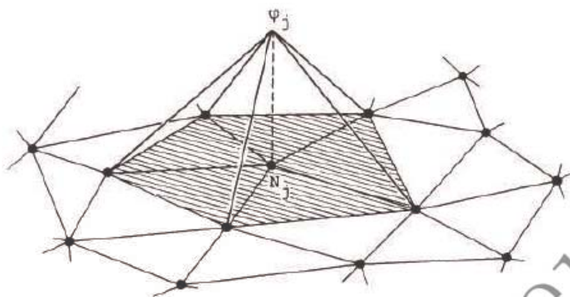

Figure 1: An example of piecewise-linear "hat" basis function in 2D

A function $u^h(x, y) \in S_0^h(\Omega)$ now has the representation: (See Figure 2)

$$u^h(x, y) = \sum_{j=1}^{M} u_j^h \varphi_j(x, y), u_j^h = u^h(N_j).$$

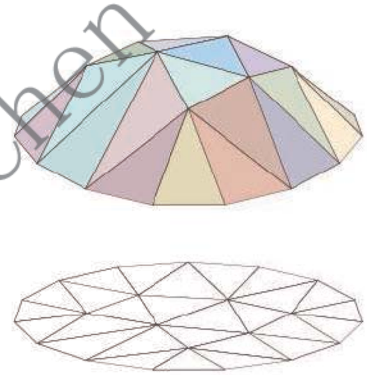

Figure 2: An example of piecewise linear function in 2D

Now we can formulate the finite element approximation for problem (1) from weak formulation (2) as: seek $u^h \in S_0^h(\Omega)$ such that

$$A(u^h, v^h) = (f, v^h),\ \ \forall v^h \in S_0^h(\Omega). \tag{3}$$

Let $u^h(x,y) = \sum_{j=1}^{M} U_j^h \varphi_j(x,y)$, $v^h(x,y) = \varphi_i(x,y)$ and substitute into (3). The result is a linear system of M equations with M unknowns $\{U_j^h\}_{j=1}^M$; i.e.,

$$\mathbf{A}\mathbf{U} = \mathbf{F}. \tag{4}$$

where $\mathbf{A} = \big(A(\varphi_j, \varphi_i)\big)_{M\times M}, \mathbf{U} = \big(U_j^h\big)_{M\times 1}, \mathbf{F} = \big((f, \varphi_i)\big)_{M\times 1}$.

The matrix $\mathbf{A}$ is referred to as the stiffness matrix, and easy to show that it is symmetric and positive definite, so the linear system (4) has a unique solution. We can use direct methods, like Cholesky decomposition method; or iterative methods, like conjugate gradient method or multigrid method to solve this system and get the finite element approximate solution of problem (1).

b. The factors that affect the accuracy of finite element methods for the approximate solution of this problem are as followed:

1. Accuracy of the linear system solver;
2. Accuracy of the quadrature rules for the integration;
3. Regularity level of the triangulation (Delaunay triangulation, regular triangulation);
4. Level of triangulation refinement (mesh size);
5. Level of approximation of finite dimensional space S_0^h to the space H_0^1 (choice of the basis functions);
6. Level of the segment approximate to the (curve) boundary;
7. Existence, uniqueness and stability for the exact solution of the problem;
8. Smoothness for the exact solution of the problem.

Specifically, let $u \in H_0^1(\Omega) \cap H^2(\Omega)$ (this can be guaranteed if $f \in L^2(\Omega)$) and u^h be the Galerkin approximation of u in the space $S_0^h(\Omega)$, i.e., piecewise-linear function u^h satisfies (3). Then there exists a positive constant C, independent of u, h, or u^h such that

$$||u - u^h||_0 \le Ch||u||_1, \tag{5}$$

$$||u - u^h||_0 \le Ch^2||u||_2, \tag{6}$$

$$||u - u^h||_1 \le Ch||u||_2. \tag{7}$$

It is important to note that if the solution u to problem (1) is not smooth enough, i.e., $u \in H_0^1(\Omega)$ but $u \notin H^2(\Omega)$, then (6) and (7) do not hold, in this case only linear convergence rate can be obtained, one order lower than the optimal rate. This is a consequence of the approximation theory, not an artifact of our finite element analysis. But it is the most important factor (the limitation) for all FEM.

2 Spring 2009

Given the functions $f(x, y)$ and $g(x, y)$, consider the problem:

$$\begin{cases} -\dfrac{\partial^2 u}{\partial x^2} - \dfrac{\partial^2 u}{\partial y^2} + g(x,y)u = f(x,y), & \text{for } 0 < x, y < 1 \\ u(x,0) = u(x,1) = 0, & \text{for } 0 \le x \le 1 \\ u(0,y) = u(1,y) = 0, & \text{for } 0 \le y \le 1 \end{cases} \tag{8}$$

a. Discuss how you would determine an approximate solution of this problem using a piecewise-linear finite element method.

b. Discuss the factors that affect the accuracy of finite element methods for the approximation solution of this problem.

Solution:

Define $\Omega = (0,1) \times (0,1), \Gamma = \partial\Omega$, seek $u \in H_0^1(\Omega) \equiv \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma\}$, such that

$$\begin{aligned} (f, v) \equiv \int_\Omega f v \mathrm{d}V &= \int_\Omega (-\Delta u + gu)\, v \mathrm{d}V \overset{\text{Green' formula}}{=\!=\!=} \int_\Omega \nabla u \cdot \nabla v \mathrm{d}V - \int_\Gamma \frac{\partial u}{\partial n} v \mathrm{d}S + \int_\Omega guv \mathrm{d}V \\ &= \int_\Omega \nabla u \cdot \nabla v \mathrm{d}V + \int_\Omega guv \mathrm{d}V \equiv A(u, v). \end{aligned} \tag{9}$$

In order to determine an approximation solution, we need to construct a finite dimensional subspace $S_0^h(\Omega) \subset H_0^1(\Omega)$:

First make a triangulation of Ω, by subdividing Ω into a set $T^h = \{K_1, ..., K_m\}$ of non-overlapping triangles K_i,

$$\Omega = \bigcup_{K \subset T^h} K = K_1 \cup K_2 \cup ... \cup K_m,$$

such that no vertex of one triangle lies on the edge of another triangle.

Define S_0^h as follows:

$$S_0^h(\Omega) = \{v : v \text{ is continuous on } \Omega, v|_K \text{ is linear for } K \in T_h, v = 0 \text{ on } \Gamma\}.$$

Here, $v|_K$ denotes the restriction of v to K. The subspace $S_0^h(\Omega)$ consists of all continuous functions that are linear on each triangle K and vanish on Γ. The corresponding basis functions $\varphi_j \in S_0^h(\Omega),\ \ j = 1, ..., M$ are then defined by:

$$\varphi_j(N_i) = \delta_{ij} \equiv \begin{cases} 1, & i = j; \\ 0, & i \neq j. \end{cases}$$

Here $N_i,\ \ i = 1, ..., M$ are inner nodes of domain Ω.

Now we can formulate the finite element approximation for problem (8) from weak formulation (9) as: seek $u^h \in S_0^h(\Omega)$ such that

$$A(u^h, v^h) = (f, v^h),\ \ \forall v^h \in S_0^h(\Omega). \tag{10}$$

Let $u^h(x, y) = \sum_{j=1}^{M} U_j^h \varphi_j(x, y)$, $v^h(x, y) = \varphi_i(x, y)$ and substitute into (10). The result is a linear system of M equations with M unknowns $\{U_j^h\}_{j=1}^M$; i.e.,

$$\mathbf{AU} = \mathbf{F}. \tag{11}$$

where $\mathbf{A} = \big(A(\varphi_j, \varphi_i)\big)_{M\times M}, \mathbf{U} = \big(U_j^h\big)_{M\times 1}, \mathbf{F} = \big((f, \varphi_i)\big)_{M\times 1}$.

The matrix $\mathbf{A}$ is symmetric and positive definite (when $g(x, y) \geq 0$), so the linear system (11) has a unique solution. We can use direct methods, like Cholesky decomposition method; or iterative methods, like conjugate gradient method or multigrid method to solve this system and get the finite element approximate solution of problem (8).

b. The factors that affect the accuracy of finite element methods for the approximate solution of this problem are the same as we described in Spring 2008.

3 Summer 2009

Consider the two point boundary value problem (BVP)

$$\begin{cases} -\dfrac{\mathrm{d}^2u}{\mathrm{d}x^2} + p\dfrac{\mathrm{d}u}{\mathrm{d}x} + qu = f(x), & \text{for } a < x < b \\ u(a) = 0, & \\ \alpha u(b) + u'(b) = 1. & \end{cases} \tag{12}$$

where p, q, α are scalars.

a. Write down a weak formulation of this problem. Show that a solution to this classical two point BVP is also a solution of your weak problem. Is the converse always true? Why or why not?

b. Suppose we want to approximate the solution of the weak problem using continuous, piecewise linear polynomials defined over a uniform partition $x_j, j = 0, ..., n+1$ of $[a, b]$ where $x_0 = a, x_{n+1} = b$. Write a discrete weak problem.

c. Assume that we use the standard "hat" basis functions. Show that once the basis functions are chosen we can write the discrete weak problem as a linear system. If $p = q = \alpha = 0$, what are the properties of this linear system? Explicitly determine the entries of the coefficient matrix when $p = q = \alpha = 0$ in this linear system assuming we use the midpoint rule to compute the entries.

d. Discuss the rates of convergence in both the H^1 and L^2 norms that you expect using continuous, piecewise linear polynomials.

Solution:

a. The weak formulation of problem (12) is as followed:

Seek $u \in V \equiv \{v | v \in H^1(a, b), v(a) = 0\}$, such that

$$A(u, v) = F(v), \ \ \forall v \in V. \tag{13}$$

where $A(u, v) \equiv \displaystyle\int_a^b (u'v' + pu'v + quv)\mathrm{d}x + \alpha u(b)v(b), \ \ F(v) \equiv \int_a^b fv\mathrm{d}x + v(b).$

Note that if $u(x)$ is the classical solution of (12), then $u(x)$ also satisfies the weak problem because for $v \in V$

$$
\begin{aligned}
F(v) \equiv \int_a^b f v \mathrm{d}x + v(b) &= \int_a^b (-u'' + pu' + qu) v \mathrm{d}x + v(b) \\
&= \int_a^b (u'v' + pu'v + quv) \mathrm{d}x + u'(a)v(a) - u'(b)v(b) + v(b). \quad (14) \\
&= \int_a^b (u'v' + pu'v + quv) \mathrm{d}x + \alpha u(b)v(b) \equiv A(u,v).
\end{aligned}
$$

Conversely, if $u \in V$ satisfied (13) and if u is suffeciently smooth, i.e., $u \in C^2(a,b)$, a situation which can be guaranteed if f is sufficiently smooth, then u coincides with the classical solution of (12). The homogeneous Dirichlet boundary conditions $u(a) = 0$ are satisfied because $u \in V$ and the differential equation holds because

$$
\begin{aligned}
A(u,v) - F(v) &= \int_a^b (u'v' + pu'v + quv) \mathrm{d}x + \alpha u(b)v(b) - \Big(\int_a^b f v \mathrm{d}x + v(b) \Big) \\
&= \int_a^b (-u'' + pu' + qu - f) v \mathrm{d}x + \Big(u'(b) + \alpha u(b) - 1 \Big) v(b) - u'(a)v(a) \\
&= 0. \qquad \forall v \in V
\end{aligned}
\quad (15)
$$

Since (15) holds for arbitrary v, in particular for $v(b) = 0$, we get

$$
-u'' + pu' + qu - f = 0 \quad (16)
$$

Thus (15) is reduced to

$$
\Big((u'(b) + \alpha u(b) - 1 \Big) v(b) = 0 \quad (17)
$$

Again since (17) holds for arbitrary v, we get

$$
u'(b) + \alpha u(b) - 1 = 0 \quad (18)
$$

However, if u is not sufficiently smooth, i.e., $u \notin C^2(a,b)$, then the converse is not true.

b. Choose $S^h \subset V$ to be the space of continuous piecewise linear polynomials defined over a uniform partition $x_j, j = 0, ..., n+1$ of $[a,b]$ where $x_0 = a, x_{n+1} = b$, which satisfy the homogeneous Dirichlet boundary condition $u(a) = 0$. Then the discrete weak problem of (13) is as followed:

Seek $u^h \in S^h$, such that

$$
A(u^h, v^h) = F(v^h), \ \forall v^h \in S^h. \quad (19)
$$

c. Consider the standard “hat” basis functions:

$$\varphi_j(x) = \begin{cases} \dfrac{x - x_{j-1}}{x_j - x_{j-1}}, & x \in [x_{j-1}, x_j]; \\ \dfrac{x_{j+1} - x}{x_{j+1} - x_j}, & x \in [x_j, x_{j+1}]; \qquad (j = 1, ..., n) \\ 0, & \text{elsewhere.} \end{cases} \tag{20}$$

$$\varphi_{n+1}(x) = \begin{cases} \dfrac{x - x_n}{x_{n+1} - x_n}, & x \in [x_n, x_{n+1}]; \\ 0, & \text{elsewhere.} \end{cases} \tag{21}$$

Clearly $\varphi_j(x) \in S^h(a, b)$ for $1 \le j \le n+1$. Moreover,

$$\varphi_j(x_i) = \delta_{ij} \equiv \begin{cases} 1, & \text{if} \quad i = j; \\ 0, & \text{otherwise.} \end{cases} \tag{22}$$

for $1 \le j \le n+1$ and $0 \le i \le n+1$ and it is easy to justify that the function defined in (20) and (21) form a basis for $S^h(a, b)$.

Since $u^h(x) \in S^h(a, b)$, let $u^h(x) = \sum_{j=1}^{n+1} U_j^h \varphi_j(x)$, $v^h(x) = \varphi_i(x)$ and substitute into (19). The result is a linear system of $n+1$ equations with $n+1$ unknowns $\{U_j^h\}_{j=1}^{n+1}$; i.e.,

$$\mathbf{A}\mathbf{U} = \mathbf{F}. \tag{23}$$

where $\mathbf{A} = \big(A(\varphi_j, \varphi_i)\big)_{(n+1)\times(n+1)}$, $\mathbf{U} = \big(U_j^h\big)_{(n+1)\times 1}$, $\mathbf{F} = \big(F(\varphi_i)\big)_{(n+1)\times 1}$.

If $p = q = \alpha = 0$, $A(u, v) = \int_a^b u'v'\mathrm{d}x$, let $h = x_{j+1} - x_j$ $(j = 0, ..., n)$, then by using the midpoint rule in each element to evaluate the integrals, the entries of $\mathbf{A}$ are given explicitly by

$$A_{ij}^h = A^h(\varphi_j, \varphi_i) \equiv \sum_{k=1}^{n+1} h\varphi_j'(\frac{x_{k-1} + x_k}{2})\varphi_i'(\frac{x_{k-1} + x_k}{2}) \tag{24}$$

In our case, $A^h = A$, recall that we have chosen $S^h(a, b)$ as the space of continuous piecewise linear functions and thus the integrands in A are constant on each element T_k, the midpoint rule integrates constant functions exactly even though we are implementing a quadrature rule, we have performed the integrations exactly, so the entries of A can be computed as followed:

$$
\begin{aligned}
A(\varphi_j,\varphi_j)=\int_a^b \varphi_j'\varphi_j'\mathrm{d}x &= \int_{x_{j-1}}^{x_j} \varphi_j'^2\mathrm{d}x+\int_{x_j}^{x_{j+1}} \varphi_j'^2\mathrm{d}x \\
&= \int_{x_{j-1}}^{x_j} (\frac{1}{h})^2\mathrm{d}x+\int_{x_j}^{x_{j+1}} (-\frac{1}{h})^2\mathrm{d}x \quad (j=1,...,n+1) \\
&= \frac{2}{h}
\end{aligned} \tag{25}
$$

$$
\begin{aligned}
A(\varphi_j,\varphi_{j+1})=A(\varphi_{j+1},\varphi_j)=\int_a^b \varphi_j'\varphi_{j+1}'\mathrm{d}x &= \int_{x_j}^{x_{j+1}} \varphi_j'\varphi_{j+1}'\mathrm{d}x \\
&= \int_{x_j}^{x_{j+1}} (\frac{1}{h})(-\frac{1}{h})\mathrm{d}x \quad (j=1,...,n) \\
&= -\frac{1}{h}
\end{aligned} \tag{26}
$$

So

$$
A=\frac{1}{h}\begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \ddots & \vdots \\ 0 & -1 & 2 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -1 \\ 0 & \cdots & 0 & -1 & 2 \end{pmatrix}. \tag{27}
$$

And it is easy to prove that the matrix A is symmetric, positive definite and tridiagonal.

d. let $u \in V \cap H^2(a,b)$ (this can be guaranteed if $f \in L^2(a,b)$) and u^h be the Galerkin approximation of u in the space $S^h(a,b)$, i.e., piecewise-linear function u^h satisfies (13). Then there exists a positive constant C, independent of u, h, or u^h such that

$$||u-u^h||_0 \le Ch^2||u||_2, \tag{28}$$

$$||u-u^h||_1 \le Ch||u||_2. \tag{29}$$

i.e., the rate of convergence in the L^2 norms is $O(h^2)$, which is quadratic convergence, and in the H^1 norms is $O(h)$, which is linear convergence.

4 Spring 2010

Consider the diffusion equation

$$u_t = \alpha u_{xx} \tag{30}$$

with the initial and boundary conditions

$$u(x,0) = g(x), \;\; u(0,t) = u_L, \;\; u(1,t) = u_R. \tag{31}$$

The function $g(x)$ is prescribed over the interval $0 < x < 1$, and α, u_L and u_R are constants and $\alpha > 0$.

a. The backward-time difference scheme can be used to convert the above initial-boundary value problem into a two-point boundary value problem (BVP) at every time step. Carry out the details of this step and develop this BVP. (20%)

b. Develop a complete piecewise-linear Galerkin-type finite element scheme to solve the resulting boundary value problem derived in part (a). (70%)

c. Comment on the numerical stability of the backward-time finite element scheme developed in (a) and (b) above. (10%)

Solution:

a. Let $\Delta t_n = t^n - t^{n-1} (n \geq 1)$ and $t^0 = 0$, by using the backward-time difference scheme to approximate u_t term, we get

$$u_t(x, t^n) \approx \frac{u(x, t^n) - u(x, t^{n-1})}{\Delta t_n} \tag{32}$$

Replacing the term u_t by this difference quotient in (30), we get

$$\frac{u(x, t^n) - u(x, t^{n-1})}{\Delta t_n} = \alpha u_{xx}(x, t^n) \tag{33}$$

$$u(x, t^n) - \alpha \Delta t_n u_{xx}(x, t^n) = u(x, t^{n-1}) \tag{34}$$

where $n \geq 1$, since for each time step, $u(x, t^{n-1})$ is known, then (34) with the boundary condition $u(0, t^n) = u_L$ and $u(1, t^n) = u_R$ is a two-point boundary value problem (BVP), converted from the above initial-boundary value problem at every time step.

b. Let $U(x) = u(x, t^n)$, the resulting boundary value problem derived in part (a) is as followed:

$$\begin{cases} U(x) - \alpha\Delta t_n U_{xx}(x) &= u(x, t^{n-1}), \quad 0 < x < 1 \\ U(0) &= u_L, \\ U(1) &= u_R. \end{cases} \tag{35}$$

Let $p(x)$ be a sufficiently smooth function where $p(0) = u_L, p(1) = u_R$ and $w(x) = U(x) - p(x)$, then the problem (35) convert to the following problem:

$$\begin{cases} w(x) - \alpha\Delta t_n w_{xx}(x) &= u(x, t^{n-1}) + \alpha\Delta t_n p_{xx}(x) - p(x), \quad 0 < x < 1 \\ w(0) &= 0, \\ w(1) &= 0. \end{cases} \tag{36}$$

The weak formulation of problem (36) is as followed:

Seek $w \in H_0^1(0,1)$, such that

$$A(w, v) = F(v) - A(p, v), \ \ \forall\, v \in H_0^1(0,1). \tag{37}$$

where $A(u, v) \equiv \int_0^1 \big(\alpha\Delta t_n u'(x)v'(x) + u(x)v(x)\big)\mathrm{d}x$, $F(v) \equiv \int_0^1 u(x, t^{n-1})v(x)\mathrm{d}x$.

Choose $S_0^h \subset H_0^1(0,1)$ to be the space of continuous piecewise linear polynomials defined over a partition $x_j, j = 0, ..., n+1$ of $[a, b]$ where $x_0 = 0, x_{n+1} = 1$, which satisfy the homogeneous Dirichlet boundary condition $u(0) = 0$ and $u(1) = 0$. Then the discrete weak problem of (37) is as followed:

Seek $w^h \in S_0^h$, such that

$$A(w^h, v^h) = F(v^h) - A(p^h, v^h), \ \ \forall v^h \in S_0^h. \tag{38}$$

Consider the standard "hat" basis functions:

$$\varphi_j(x) = \begin{cases} \dfrac{x - x_{j-1}}{x_j - x_{j-1}}, & x \in [x_{j-1}, x_j]; \\ \dfrac{x_{j+1} - x}{x_{j+1} - x_j}, & x \in [x_j, x_{j+1}]; \qquad (j = 1, ..., n) \\ 0, & \text{elsewhere.} \end{cases} \tag{39}$$

Cleary $\varphi_j(x) \in S_0^h(0,1)$ for $1 \le j \le n$. Moreover,

$$\varphi_j(x_i) = \delta_{ij} \equiv \begin{cases} 1, & \text{if} \quad i = j; \\ 0, & \text{otherwise.} \end{cases} \tag{40}$$

for $1 \le j \le n$ and $0 \le i \le n+1$ and it is easy to justify that the function defined in (39) form a basis for $S_0^h(0,1)$.

Since $w^h(x) \in S_0^h(0,1)$, let $w^h(x) = \sum_{j=1}^{n} W_j^h \varphi_j(x)$, $v^h(x) = \varphi_i(x)$ and substitute into (38). The result is a linear system of n equations with n unknowns $\{W_j^h\}_{j=1}^n$; i.e.,

$$\mathbf{AW} = \mathbf{F}. \tag{41}$$

where $\mathbf{A} = \big(A(\varphi_j, \varphi_i)\big)_{n\times n}, \mathbf{W} = \big(W_j^h\big)_{n\times 1}, \mathbf{F} = \big(F(\varphi_i) - A(g, \varphi_i)\big)_{n\times 1}$.

At the end, $U^h(x) = w^h(x) + p(x)$ is the FEM solution by the request.

c. The backward-time finite element scheme developed in (a) and (b) above is unconditional stable, i.e., this scheme is stable regardless of the size of the time steps.

5 Spring 2011

Consider the simple 1-D diffusion equation:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad 0 < x < l, \quad t > 0 \tag{42}$$

subject to

$$u(0,t) = u(l,t) = 0, \quad t > 0 \tag{43}$$

$$u(x,0) = u_0(x), \quad 0 < x \leq l \tag{44}$$

Using Galerkin approximation with piecewise linear basis function which satisfy the boundary conditions

$$\Phi_i(0) = \Phi_i(l) = 0 \tag{45}$$

a. Show that if we use approximate solution

$$U(x,t) = \sum_{j=1}^{N} u_j(t)\Phi_j(x) \tag{46}$$

That we obtain

$$\sum_{j=1}^{N} \left\{ \frac{\mathrm{d}u_j}{\mathrm{d}t} b_{ij} + u_j g_{ij} \right\} = 0, \quad i = 1, 2, ..., N \tag{47}$$

where

$$g_{ij} = \int_0^l \frac{\mathrm{d}\Phi_i}{\mathrm{d}x}\frac{\mathrm{d}\Phi_j}{\mathrm{d}x}\mathrm{d}x, \quad i, j = 1, 2, ..., N \quad \text{(stiffness matrix)} \tag{48}$$

$$b_{ij} = \int_0^l \Phi_i \Phi_j \mathrm{d}x, \quad i, j = 1, 2, ..., N \quad \text{(mass matrix)} \tag{49}$$

i.e., a system

$$B\dot{U} + GU = b \quad (\text{ here } \quad b = 0 \ ) \tag{50}$$

where

$$B = (\Phi_j, \Phi_i) \tag{51}$$

$$G = a(\Phi_j, \Phi_i) \tag{52}$$

b. Show that if h is the element size, we have that matrices B and G have the following entries

$$B = \frac{h}{6} \begin{bmatrix} 4 & 1 & & & & & \\ 1 & 4 & 1 & & & & \\ & \cdot & \cdot & \cdot & & & \\ & & \cdot & \cdot & \cdot & & \\ & & & \cdot & \cdot & \cdot & \\ & & & & 1 & 4 & 1 \\ & & & & & 1 & 4 \end{bmatrix}. \tag{53}$$

$$G = \frac{1}{h}\begin{bmatrix} 2 & -1 & & & & & \\ -1 & 2 & -1 & & & & \\ & \cdot & \cdot & \cdot & & & \\ & & \cdot & \cdot & \cdot & & \\ & & & \cdot & \cdot & \cdot & \\ & & & & -1 & 2 & -1 \\ & & & & & -1 & 2 \end{bmatrix}. \tag{54}$$

for piecewise linear hat basis functions on regular mesh.

Solution:

a. The weak formulation of problem (42) is as followed:

For a fixed time T, seek $U \in L_2(0,T;H_0^1(0,l)), U_t \in L_2(0,T;H^{-1}(0,l))$ and $U(x,0) = u_0(x)$, such that

$$(U_t, v) + (U_x, v_x) = 0, \quad \forall v \in H_0^1(0,l) \tag{55}$$

Let $U(x,t) = \sum_{j=1}^{N} u_j(t)\Phi_j(x)$, $v = \Phi_i(x)$ and replace them to the equation (55), we get

$$\int_0^l \sum_{j=1}^{N} \left\{ \frac{\mathrm{d}u_j}{\mathrm{d}t}\Phi_j\Phi_i + u_j \frac{\mathrm{d}\Phi_j}{\mathrm{d}x}\frac{\mathrm{d}\Phi_i}{\mathrm{d}x} \right\} = 0, \quad i = 1,2,...,N \tag{56}$$

So we obtain

$$\sum_{j=1}^{N} \left\{ \frac{\mathrm{d}u_j}{\mathrm{d}t} b_{ij} + u_j g_{ij} \right\} = 0, \quad i = 1,2,...,N \tag{57}$$

where

$$g_{ij} = \int_0^l \frac{\mathrm{d}\Phi_i}{\mathrm{d}x}\frac{\mathrm{d}\Phi_j}{\mathrm{d}x}\mathrm{d}x, \quad i,j = 1,2,...,N \quad \text{(stiffness matrix)} \tag{58}$$

$$b_{ij} = \int_0^l \Phi_i\Phi_j \mathrm{d}x, \quad i,j = 1,2,...,N \quad \text{(mass matrix)} \tag{59}$$

i.e., a system

$$B\dot{U} + GU = b \quad (\text{ here } \quad b = 0 \ ) \tag{60}$$

where

$$B = (\Phi_j, \Phi_i) \tag{61}$$

$$G = a(\Phi_j, \Phi_i) \tag{62}$$

b.

$$
\begin{aligned}
B(\varphi_j,\varphi_j)=\int_0^l \varphi_j^2 \mathrm{d}x &= \int_{x_{j-1}}^{x_j} \varphi_j^2 \mathrm{d}x + \int_{x_j}^{x_{j+1}} \varphi_j^2 \mathrm{d}x \\
&= \int_{x_{j-1}}^{x_j} (\frac{x-x_{j-1}}{h})^2 \mathrm{d}x + \int_{x_j}^{x_{j+1}} (\frac{x_{j+1}-x}{h})^2 \mathrm{d}x \quad (j=1,...,N) \\
&= \frac{2}{3}h
\end{aligned}
\tag{63}
$$

$$
\begin{aligned}
B(\varphi_j,\varphi_{j+1})=B(\varphi_{j+1},\varphi_j)=\int_0^l \varphi_j\varphi_{j+1} \mathrm{d}x &= \int_{x_j}^{x_{j+1}} \varphi_j\varphi_{j+1} \mathrm{d}x \\
&= \int_{x_j}^{x_{j+1}} (\frac{x-x_j}{h})(\frac{x_{j+1}-x}{h}) \mathrm{d}x \quad (j=1,...,N-1) \\
&= \frac{1}{6}h
\end{aligned}
\tag{64}
$$

So

$$
B=\frac{h}{6}\begin{bmatrix} 4 & 1 & & & & \\ 1 & 4 & 1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & \ddots & \ddots \\ & & & 1 & 4 & 1 \\ & & & & 1 & 4 \end{bmatrix}. \tag{65}
$$

$$
\begin{aligned}
G(\varphi_j,\varphi_j)=\int_0^l \varphi_j'^2 \mathrm{d}x &= \int_{x_{j-1}}^{x_j} \varphi_j'^2 \mathrm{d}x + \int_{x_j}^{x_{j+1}} \varphi_j'^2 \mathrm{d}x \\
&= \int_{x_{j-1}}^{x_j} (\frac{1}{h})^2 \mathrm{d}x + \int_{x_j}^{x_{j+1}} (-\frac{1}{h})^2 \mathrm{d}x \quad (j=1,...,N) \\
&= \frac{2}{h}
\end{aligned}
\tag{66}
$$

$$
\begin{aligned}
G(\varphi_j,\varphi_{j+1})=G(\varphi_{j+1},\varphi_j)=\int_0^l \varphi_j'\varphi_{j+1}' \mathrm{d}x &= \int_{x_j}^{x_{j+1}} \varphi_j'\varphi_{j+1}' \mathrm{d}x \\
&= \int_{x_j}^{x_{j+1}} (\frac{1}{h})(-\frac{1}{h}) \mathrm{d}x \quad (j=1,...,N-1) \\
&= -\frac{1}{h}
\end{aligned}
\tag{67}
$$

So

$$G = \frac{1}{h}\begin{bmatrix} 2 & -1 & & & & & \\ -1 & 2 & -1 & & & & \\ & \cdot & \cdot & \cdot & & & \\ & & \cdot & \cdot & \cdot & & \\ & & & \cdot & \cdot & \cdot & \\ & & & & -1 & 2 & -1 \\ & & & & & -1 & 2 \end{bmatrix}. \tag{68}$$

6 Summer 2011

a. Let Ω be a bounded domain in R^2 with boundary $\Gamma = \Gamma_1 \cup \Gamma_2$ where $\Gamma_1 \cap \Gamma_2 = \emptyset$. Consider the following PDE and boundary conditions for $u(x, y)$

$$-\Delta u + uu_x = f(x,y), \quad (x,y) \in \Omega \tag{69}$$

$$u = 0 \quad \text{on} \quad \Gamma_1 \qquad \frac{\partial u}{\partial n} = 4 \quad \text{on} \quad \Gamma_2 \tag{70}$$

and the weak formulation

Seek $u \in \hat{H}^1$ satisfying

$$\int_\Omega \nabla u \cdot \nabla v + \int_\Omega uu_x v = \int_\Omega fv + 4\int_{\Gamma_2} v, \quad \forall v \in \hat{H}^1 \tag{71}$$

where $\hat{H}^1$ is all functions that are zero on Γ_1 and which possess one weak derivative. Here $\Delta u = u_{xx} + u_{yy}$ and $\partial u/\partial n$ denotes the derivative of u in the direction of the unit outer normal, i.e., $\nabla u \cdot \vec{n}$. Show that if u satisfies the classical boundary value problem then it satisfies the weak problem. Then show that if u is a sufficiently smooth solution to the weak problem, then it satisfies the PDE and the boundary conditions.

Solution:

If u satisfies the classical boundary value problem (69), multiplied by an appropriate test function v, integrating over the domain, we get

$$\int_\Omega \big(-\Delta u + uu_x \big) v = \int_\Omega fv, \quad v \in \hat{H}^1 \tag{72}$$

then use the Green's formula, we get

$$\int_\Omega \nabla u \cdot \nabla v + \int_\Omega uu_x v = \int_\Omega fv + \int_{\Gamma_2} \frac{\partial u}{\partial n} v, \quad \forall v \in \hat{H}^1 \tag{73}$$

replacing the boundary condition $\dfrac{\partial u}{\partial n} = 4$ on Γ_2 in (73), we get

$$\int_\Omega \nabla u \cdot \nabla v + \int_\Omega uu_x v = \int_\Omega fv + 4\int_{\Gamma_2} v, \quad \forall v \in \hat{H}^1 \tag{74}$$

Conversely, if u is a sufficiently smooth solution to the weak problem, by using the Green's formula

$$\begin{aligned} & \int_\Omega \nabla u \cdot \nabla v + \int_\Omega uu_x v - \int_\Omega fv - 4\int_{\Gamma_2} v \\ = \; & \int_\Omega \big(-\Delta u + uu_x - f \big) v + \int_{\Gamma_2} \big(\frac{\partial u}{\partial n} - 4 \big) v = 0 \quad \forall v \in \hat{H}^1 \end{aligned} \tag{75}$$

Since (75) holds in particular for all $v = 0$ on Γ, so

$$-\Delta u + uu_x = f \tag{76}$$

Thus (75) is reduced to

$$\int_{\Gamma_2} (\frac{\partial u}{\partial n} - 4)v = 0 \tag{77}$$

Now varying v over H^1, which means that v will vary freely on Γ_2, we finally get

$$\frac{\partial u}{\partial n} = 4 \quad \text{on} \quad \Gamma_2 \tag{78}$$

then u satisfies the PDE and the boundary conditions.