Finite Element Treatment of the Navier Stokes Equations, Part V

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1 Introduction

2 Evaluating The Strong Navier-Stokes Equations

Now that we have some idea of the geometry of our region, and how the basis functions are generated, we are ready to use these ideas to treat the Navier-Stokes equations. We begin with the classical form of the equation.

Suppose we are interested in the particular form of the Navier Stokes equations represented by time-independent 2D incompressible flow, as discussed earlier. Let us suppose that we have an arbitrary pair of functions $(\mathbf{v}, p)$. We say this pair of functions solves (our version of) the Navier Stokes Equations *in the strong sense*, or is a *strong solution* if

$$-\Delta\mathbf{v}(x,y) + R \quad (\mathbf{v}(x,y)\cdot\nabla)\mathbf{v}(x,y) + \nabla p(x,y) = 0 \tag{1}$$
$$\nabla\mathbf{v}(x,y) = 0 \quad \forall(x,y)\in\Omega \tag{2}$$

Yes, a strong solution is really what we would usually just call a solution; we only mention the term strong solution to distinguish it from the kind of solutions we're actually going to be able to compute, which are termed "weak solutions".

3 The Weak Form of the Navier-Stokes Equations

If a pair of functions is a strong solution to the Navier Stokes equations, then it satisfies those equations pointwise. Certainly, this pair will still be a solution if we multiply the Navier Stokes equations by any functions we like. In particular, we could multiply the momentum equation by any velocity basis function $\psi_i(x,y)$ and the continuity equation by any pressure basis function $\phi_j(x,y)$. A strong solution will surely satisfy:

$$(-\Delta\mathbf{v}(x,y) + R(\mathbf{v}(x,y)\cdot\nabla)\mathbf{v}(x,y) + \nabla p(x,y))\psi_i(x,y) = 0 \tag{3}$$
$$\nabla\mathbf{v}(x,y)\phi_j(x,y) = 0 \tag{4}$$
$$\forall(x,y)\in\Omega \tag{5}$$

We could take a step further and integrate these equations over the domain. For any strong solution, it must be the case that

$$\int_\Omega (-\Delta\mathbf{v}(x,y) + R(\mathbf{v}(x,y)\cdot\nabla)\mathbf{v}(x,y) + \nabla p(x,y))\psi_i(x,y)d\Omega = 0 \tag{6}$$

$$\int_\Omega \nabla\mathbf{v}(x,y)\phi_j(x,y)d\Omega = 0 \tag{7}$$

(To cut down on the length of the equations, we are now going to suppress the explicit dependence of quantities on x and y.)

4 Lowering the Order using Integration by Parts

We are going to apply integration by parts to modify this equation a little further. Consider, then, the following:

$$\frac{d}{dx}(\frac{dv}{dx}\psi_i) = \frac{d^2v}{dx^2}\psi_i + \frac{dv}{dx}\frac{d\psi_i}{dx} \tag{8}$$

If we integrate and rearrange, we have

$$\int_\Omega -\frac{d^2v}{dx^2}\psi_i d\Omega = -\frac{d}{dx}(\frac{dv}{dx}\psi_i)|_{\partial Omega} + \int_\Omega \frac{dv}{dx}\frac{d\psi_i}{dx}d\Omega \tag{9}$$

Now if we suppose ψ_i is a basis function whose associated node is not on the boundary, then ψ_i is identically zero on the boundary. Therefore, the boundary term disappears, and we have

$$\int_\Omega -\frac{d^2v}{dx^2}\psi_i d\Omega = \int_\Omega \frac{dv}{dx}\frac{d\psi_i}{dx}d\Omega \tag{10}$$

Repeating this process for the y derivative, we then have

$$\int_\Omega -\Delta\mathbf{v}\psi_i d\Omega = \int_\Omega \frac{d\mathbf{v}}{dx}\frac{d\psi_i}{dx} + \frac{d\mathbf{v}}{dy}\frac{d\psi_i}{dy}d\Omega \tag{11}$$

and hence we can rewrite the momentum equation as

$$\int_\Omega \nabla\mathbf{v}\cdot\nabla\psi_i + (R(\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla p)\psi_i d\Omega \quad = \quad 0 \tag{12}$$

$$\int_\Omega \nabla\mathbf{v}\phi_j d\Omega \quad = \quad 0 \tag{13}$$

Why is this an accomplishment? Well, we have lowered the order of differention on $\mathbf{v}$. That means our weak form of the equation now only requires that $\mathbf{v}$ be a $C^1(\Omega)$ function. This allows us to consider a larger class of solutions than the classical strong form of the equation would require.

We have done this at the cost of assuming that the basis functions are differentiable, but in fact, they are polynomials, so that's not a problem at all.

This, then, is our weak form of the Navier Stokes equations. Notice that, up to this point, we have not explicitly used anything from finite elements. The appearance of the function $\psi_i(x, y)$ should suggest what is on our minds, but so far, everything we have said is true for arbitrary sufficiently smooth functions $\mathbf{v}$, p, ψ_i and ϕ_i.

5 The Discretized Weak Equation

But now, let us consider the idea that, instead of looking for any functions $(\mathbf{v}, p)$ whatsoever, we might reasonably look for a solution in the finite dimensional space spanned by the velocity and pressure basis functions. A solution from this

space can be specified by $2 * N_u + N_p$ coefficients; that is, of course, exactly how many distinct equations we could get by multiplying the (vector) momentum equation by every possible velocity basis function, and the continuity equation by every possible pressure basis function.

The process of multiplying the equations you are trying to satisfy by a set of test functions, and then integrating, is called the *Galerkin method*. What the method is actually doing is requiring that the error be orthogonal to the space spanned by the test functions. Here, our "dot product" is simply the L2 inner product

$$< u, v > \equiv \int_\Omega u(x, y)v(x, y)d\Omega \tag{14}$$

When, as in our case, the test functions are also the set of basis functions used to represent solutions, the process is called the *Petrov-Galerkin method*.

So far, we have not worried much about the boundary conditions...

6 The Nonlinear System for the Finite Element Coefficients

For a problem like the Poisson equation, we would be done now. The Petrov-Galerkin method, along with the application of boundary conditions, results in a linear system for the finite element coefficients of the form $K * c = f$. However, the Navier Stokes equations are nonlinear, and the finite element form of them inherits this property.